ON THE LOCATION OF THE ZEROS OF THE DERIVATIVES
OF A POLYNOMIAL SYMMETRIC IN THE ORIGIN

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If the zeros of a polynomial \( p(z) \) when plotted in the \( z \)-plane are symmetric in \( 0: z = 0 \), the zeros of the derivative \( p'(z) \) of \( p(z) \) can profitably be studied by transforming onto the \( w \)-plane, with \( w = z^2 \), and applying known theorems there.\(^1\) It is the purpose of the present note to carry that study somewhat farther than has been previously done, in particular to consider the higher derivatives of \( p(z) \).

Under the transformation \( w = u + iv = z^2 = (x + iy)^2 \), an arbitrary line \( Au + Bv + C = 0 \) in the \( w \)-plane corresponds to an equilateral hyperbola \( A(x^2 - y^2) + 2Bxy + C = 0 \) in the \( z \)-plane with center 0 or to two perpendicular lines intersecting at 0. A half-plane in the \( w \)-plane for which \( w = 0 \) is an interior or exterior point corresponds in the \( z \)-plane respectively to the exterior or interior of an equilateral hyperbola whose center is 0; a half-plane for which \( w = 0 \) is a boundary point corresponds to a double sector with vertex \( z = 0 \) and angle \( \pi/2 \). A point \( z \) is considered to be exterior or interior to a hyperbola according as the curve at its nearest point is convex or concave toward \( z \).

We write the given polynomial in the form

\[
\phi(z) = z^1 \prod_{i=1}^{q} \left( z^2 - \alpha_i \right), \quad \alpha_i \neq 0,
\]

and in the \( w \)-plane study the polynomials (\( w = z^2 \))

\[
P(w) = P(z^2) = \left[ \phi(z) \right]^2, \quad P'(w) = \phi(z) \cdot \phi'(z)/z.
\]

Each zero of \( P(w) \) corresponds to a zero of \( \phi(z) \) and reciprocally; each zero of \( P'(w) \) corresponds to a zero of \( \phi(z) \) or \( \phi'(z) \) and reciprocally except that \( z = 0 \) is a zero of \( \phi'(z) \) unless \( z = 0 \) is a simple zero of \( \phi(z) \).

We have (loc. cit.) by Lucas' Theorem

**Theorem 1.** If the zeros of \( \phi(z) \) are symmetric in 0 and lie in the closed exterior of an equilateral hyperbola with center 0 or in the closed exterior of a double sector with vertex 0 and angle \( \pi/2 \), then the zeros of \( \phi'(z) \) lie also in that closed exterior.

If the zeros of \( \phi(z) \) are symmetric in 0 and lie in the closed interior of an equilateral hyperbola with center 0, then the zeros of \( \phi'(z) \) also lie in that closed interior except for a simple zero at 0.

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\(^1\) Walsh, Mathematica vol. 8 (1933) pp. 185-190.
To the polynomial $P(w)$ we now apply the theorem:\textsuperscript{2}

Let $P(w)$ be a polynomial in $w$ of degree $n$ with an $l$-fold ($l > 0$) zero at $w = 0$ and all the remaining zeros of $P(w)$ in the closed half-plane $\Pi$ not containing $w = 0$. Then except for a zero at $w = 0$ of multiplicity $l - 1$, all zeros of $P'(w)$ lie in the closed half-plane obtained by shrinking $\Pi$ toward the origin in the ratio $n : l$.

The degrees of $p(z)$ and $P(w)$ defined by (1) and (2) are equal, as are the multiplicities of their zeros $z = 0$ and $w = 0$. Under the transformation $w = z^2$, the line $u = a$ corresponds to the hyperbola $x^2 - y^2 = a$, and the line $u = la/n$ corresponds to $x^2 - y^2 = la/n$, so we have

**Theorem 2.** Let $p(z)$ be a polynomial of degree $n$ whose zeros are symmetric in the origin $0$, let $0$ be an $l$-fold zero ($l > 0$), and let all the other zeros lie in the closed interior of an equilateral hyperbola $H$ with center $0$. Then except for an $(l - 1)$-fold zero at $0$, all zeros of $P'(z)$ lie in the closed interior of the hyperbola obtained by shrinking $H$ toward $0$ in the ratio $(n : l)^{1/2}$.

We turn now to the higher derivatives of the polynomial $p(z)$. Under the conditions of the first part of Theorem 1, the zeros of every derivative of $p(z)$ not vanishing identically lie in the given closed region.

Suppose, however, the zeros of the polynomial $p(z)$ of degree $n$ lie in the closed interior of an equilateral hyperbola $H$ whose center is $0$. The polynomial $p'(z)$ has a simple zero at $0$, and by Theorem 1 the remaining zeros of $p'(z)$ lie in the closed interior of $H$. By Theorem 2, all zeros of $p''(z)$ lie in the closed interior of the hyperbola obtained by shrinking $H$ toward $0$ in the ratio $[(n - 1) : 1]^{1/2}$. The higher derivatives of $p(z)$ not vanishing identically have alternately a simple zero at $0$ and no zero at $0$. Continued application of Theorems 1 and 2, in alternation, then yields further hyperbolas which respectively contain the zeros other than $0$ of the $k$th derivative of $p(z)$. However, except in the cases $k = 1$, and $k = 2$ with $n \geq 4$, these hyperbolas are not the most favorable that can be obtained; a zero of $p'(z)$ in Theorem 2 cannot lie on the new hyperbola unless all zeros of $p(z)$ lie on $H$ in two points symmetric in $0$. We proceed to prove the following generalization of Theorem 2, the principal result of the present note:

**Theorem 3.** Let $p(z)$ be a polynomial of degree $n$ whose zeros are symmetric in $0$, let $0$ be an $l$-fold zero, and let the remaining zeros lie in the closed interior of a hyperbola $H$ whose center is $0$. Let $p_0(z)$ be the polynomial of degree $n$ whose zeros are symmetric in $0$, which has an $l$-fold

zero at 0, and whose remaining zeros lie at the vertices of $H$. Let $z_k, \ 1 \leq k \leq n-2,$ be one of the zeros of $p_0^{(k)}(z)$ of smallest positive modulus, and let $H_k$ be the equilateral hyperbola with center 0 which has $z_k$ as a vertex. Then all zeros of $p^{(k)}(z)$ other than 0 lie in the closed interior of $H_k$.

Of course 0 need not be a zero of $p^{(k)}(z)$; whether or not 0 is such a zero depends on $l$, $n$, and $k$. To prove Theorem 3 we need a more powerful result than that used in proving Theorem 2:

Let $Q(w)$ be a polynomial in $w$ of degree $q$, and let the numbers $A_j$ be constants. If the locus of the zeros of $Q(w)$ is a closed half-plane $\Pi$, then the locus of the zeros of the polynomial

\begin{equation}
A_0 w^r Q(w) + A_1 w^{r+1} Q'(w) + \cdots + A_q w^{r+q} Q^{(q)}(w)
\end{equation}

is also a number of half-planes, identical with the locus of the zeros of the polynomial (3) with $Q(w)$ replaced by $(w - \alpha)^q$:

\begin{equation}
A_0 w^r (w - \alpha)^q + q A_1 w^{r+1} (w - \alpha)^{q-1} + \cdots
\end{equation}

\begin{equation}
+ q(q - 1) \cdots 1 A_q w^{r+q}
\end{equation}

when the locus of $\alpha$ is $\Pi$.

The theorem just quoted is essentially a special case of a more general theorem\footnote{Walsh, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 163-180; Theorem 9 of that paper applies to two circular regions, here specialized to a single point and a half-plane respectively.} concerning a linear combination of products of derivatives of two polynomials. If polynomial (4) is written as the product of $w^{r+q}$ and a polynomial in $W = (w - \alpha)/w$ whose zeros are $W = W_1, W_2, \cdots, W_q$, it is seen that the common locus of the zeros of (3) and (4) consists of the origin $w = 0$ (provided $w = 0$ is a zero of (4)) plus a number of half-planes, loci of the points $w = \alpha/(1 - W_j), \ j = 1, 2, \cdots, q$, when the locus of $\alpha$ is $\Pi$; if $\Pi$ does not contain the point $w = 0$, a zero $W_j = 1$ is to be ignored; if $\Pi$ contains the point $w = 0$ and if $W_j = 1$ is a zero of (4), the entire plane is to be considered the locus of the zeros of (3) and (4). If $\Pi$ does not contain 0, and if for real positive $\alpha$ all zeros of (4) lie in the interval $0 \leq w \leq \alpha$, then the locus of the zeros of (4) is the origin (if $w = 0$ is a zero of (4)) plus a half-plane not containing 0 bounded by a line parallel to the boundary of $\Pi$, traced by the zero of (4) nearest to but different from 0 when $\alpha$ traces the boundary of $\Pi$.

In Theorem 3 we write

\begin{align*}
p(z) &= z^l p_1(z^2), \\
p_1(z) &= z^{l-1} p_1(z^2) + 2 z^{l+1} p_1'(z^2),
\end{align*}
$$p''(z) = l(l - 1)z^{l-2}p_1(z^2) + (4l + 2)z^l p'_1(z^2) + 4z^{l+2} p''_1(z^2),$$

Except perhaps for a factor $z$, these derivatives are precisely of the form (3), with $w = z^2$. If we have $p_0(z) = z_1(z^2 - \beta z) - \beta z$, the zeros of $p_0(z)$ lie on the line segment joining $\beta$ and $-\beta$, and the corresponding zeros in the $w$-plane lie on the segment joining $0$ and $\beta^2$; the images in the $w$-plane of the zeros of $p_1(z)$ lie in a half-plane $\Pi$ not containing $w = 0$; thus by the theorem quoted the images in the $w$-plane of the zeros of $p^{(k)}(z)$ other than $z = 0$ lie in the half-plane whose boundary is parallel to that of $\Pi$, which contains $\Pi$ in its interior, and whose boundary passes through the image of the zero of $p^{(k)}(z)$ nearest to but distinct from 0. Theorem 3 follows.

When as in Theorem 1 we study the zeros of a polynomial $p(z)$ that are symmetric in 0, equilateral hyperbolas with center 0 are both (1) the lines of force in the $s$-plane in the Gaussian field due to two particles symmetric in 0 and (2) the images in the $z$-plane of the straight lines in the $w$-plane under the transformation $w = z^2$. Indeed, the lines of force due to particles at $z = \alpha$ and $z = -\alpha$ are loci $\arg(z^2 - \alpha^2) = \text{const.}$, arcs of equilateral hyperbolas with center 0; half-lines in the $w$-plane can be written $\arg(w - w_0) = \text{const.}$, and their images in the $z$-plane are loci $\arg(z^2 - w_0) = \text{const.}$.

A set of points in the $z$-plane is said to have m-fold symmetry about 0 if the set is unchanged by a rotation about 0 through an angle of $2\pi/m$. A polynomial $p(z)$ whose zeros possess m-fold symmetry about 0 is readily studied by means of the transformation $w = z^m$. Under this transformation a straight line in the $w$-plane not through $w = 0$ corresponds to a curve which may be called an m-hyperbola with center 0. A closed half-plane in the $w$-plane containing 0 in its interior corresponds to the closed exterior of an m-hyperbola, and a closed half-plane not containing 0 corresponds to the closed interior of an m-hyperbola. A line through $w = 0$ corresponds to $m$ equally spaced lines through $z = 0$, and a half-plane in the $w$-plane bounded by a line through $w = 0$ corresponds to $m$ equally spaced sectors in the $z$-plane, each of angle $\pi/m$. The function $[p(w^{1/m})]^m$ is a polynomial in $w$ which is readily studied in the $w$-plane, and yields results on the zeros of $p'(z)$ in the $z$-plane. Theorem 3 extends directly to a polynomial $p(z)$ whose zeros possess m-fold symmetry in 0. The m-hyperbolas with center 0 are not merely the images in the $z$-plane of the straight lines in the $w$-plane, but also the lines of force in the $z$-plane due to $m$ particles m-fold symmetric in $z = 0$.

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