TOPOLOGICAL GROUPS AND GENERALIZED MANIFOLDS

EDWARD G. BEGLE

In a recent paper [4],

Montgomery showed that in a locally euclidean 3-dimensional group, any 2-dimensional closed subgroup is also locally euclidean. In this note we prove an analogous result for higher dimensions and more general spaces.

**THEOREM.** Let $G$ be a locally compact space which is both a topological group and an $n$-dimensional orientable generalized manifold. Let $H$ be a closed connected $(n-1)$-dimensional subgroup. Then, if $H$ carries a nonbounding $(n-1)$-cycle, $H$ is also an orientable generalized manifold.

The terminology used in the statement of this theorem, and in what follows, is that of our two previous papers on generalized manifolds [1, 2], and we assume that the reader is familiar with them.

We make, however, one change. We find it convenient to define infinite cycles in the following way: We add to $G$ an ideal point, $g^+$, taking as neighborhoods of $g^+$ those open subsets of $G$ whose closures are not compact. Then $G^+ = G \cup g^+$ is compact. Now an infinite cycle of $G$ is defined to be a relative cycle of $G^+$ mod $g^+$. That this definition of infinite cycles is equivalent to the one used in [2] follows from Theorem 1.1 of [2].

**LEMMA 1.** Given any neighborhood $M$ of the unit element $e$ of $G$, there is a neighborhood $N$ of $E$ such that for any infinite cycle $\Gamma^k$ on $H$, $0 \leq k \leq n-1$, and for any $g \in N$, $\Gamma^k \sim g \Gamma^k$ on $M \cdot H$.

**PROOF.** Let $M_{n-1} \subset M$ have a compact closure. Choose a sequence

$$M_{n-1} \supset N_{n-1} \supset M_{n-2} \supset \cdots \supset M_0 \supset N_0$$

such that $N_i$ is obtained from $M_i$ by the local connectedness of $G$ in dimension $i$, and such that $M_i, M_i \subset N_{i+1}$. Finally let $N$ be such that $N \cdot N \subset N_0$.

Now let $g \in N$. To show that $\Gamma^k \sim g \cdot \Gamma^k$ on $M \cdot H$, it is sufficient to show that the coordinates of these cycles on the nerve of any covering $U$ of $G$ are homologous on $(M \cdot H)^+$. To this end, given a covering $U$, choose $U' \triangleleft U$. Let $X$ be the complement of the union of those sets of $U'$ which contain $g^+$. Then $X$ is a compact set. Let $X_i = M_0 \cdot X$ and $X_i = M_{i-1} \cdot X_{i-1}$. Each $X_i$ is a compact set.

Received by the editors December 10, 1947.

1 Numbers in brackets refer to the bibliography at the end of the paper.
A finite number of translations of \( N_{n-1} \) cover \( X_n \), say
\[
g_{1,n-1} \cdot N_{n-1}, g_{2,n-1} \cdot N_{n-1}, \ldots, g_{k,n-1} \cdot N_{n-1}.
\]
For each \( i \), let \( U_{i,n-1} \) be a refinement of \( U' \) such that any \((n-1)\)-cycle on \( U_{i,n-1} \) in \( g_{i,n-1} \cdot N_{n-1} \) has its projection to \( U' \) bounding in \( g_{i,n-1} \cdot M_{n-1} \). Let \( U_{n-1} \) be a common refinement of these coverings.

Next, a finite number of translations of \( N_{n-2} \) cover \( X_{n-1} \). From these we obtain a refinement \( U_{n-2} \) of \( U_{n-1} \) by the procedure above, this time using the local connectedness of \( G \) in dimension \( n-2 \). Proceeding in this fashion for another \( n-2 \) steps we arrive at a covering \( U_0 \).

Let \( \Gamma_0^k \) and \( g \cdot \Gamma_0^k \) be the coordinates of \( \Gamma^k \) and \( g \cdot \Gamma^k \) on \( U_0 \). We assert that \( \pi \Gamma_0^k \) and \( \pi g \cdot \Gamma_0^k \) are homologous on \( U \) on \((M \cdot H)^+\), where \( \pi \) is the projection from \( U_0 \) to \( U \). Let \( \Delta \) be the cartesian product of \( |\Gamma_0^k| \) with a unit segment, subdivided simplicially in such a way that all the vertices of \( \Delta \) are in the base, \( \Delta_0 = |\Gamma_0^k| \times 0 \), and in the top, \( \Delta_1 = |\Gamma_0^k| \times 1 \). Let \( \overline{\Delta} \) be the closed subcomplex of \( \Delta \) generated by those simplexes of \( \Gamma_0^k \) which are on \( X \). We define a partial realization \( \tau' \) of \( \Delta \) on \( U_0 \) by letting \( \tau' \sigma = \sigma \) if \( \sigma \in \Delta_0 \) and \( \tau' \sigma = g \cdot \sigma \) if \( \sigma \in \Delta_1 \).

\( \tau' \) induces a partial realization \( \overline{\tau} \) of \( \overline{\Delta} \) on \( U_0 \). In view of the choices of the coverings made above, the usual argument shows that there is a full realization \( \overline{\tau} \) of \( \overline{\Delta} \) on \( U' \), where, if \( \pi_0 \) is the projection from \( U_0 \) to \( U' \), \( \overline{\tau} = \pi_0 \overline{\tau}' \) whenever the latter is defined. Also, \( \overline{\tau} \overline{\Delta} \) is on \( M_{n-1} \cdot H \).

We can now define a full realization of \( \Delta \) on \( U \) in the following fashion. The projection \( \pi_0 \) can be so chosen that a vertex of \( U_0 \) not on \( X \) is projected into a vertex of \( U' \) which contains \( g \). Since \( U' < U \), a projection \( \pi \) of \( U' \) to \( U \) can be so chosen that any simplex of \( U_0 \) which has a vertex not on \( X \) is projected by \( \pi \pi_0 \) into the simplex of \( U \) consisting of those vertices of \( U \) which contain \( g \). Any cycle in this simplex bounds in this simplex, so \( \pi \pi_0 \tau' (\Delta - \overline{\Delta}) \) can be filled in to make a full realization of \( \Delta - \overline{\Delta} \) on \( U \), and this together with \( \pi \overline{\tau} \overline{\Delta} \) makes a full realization \( \tau \) of \( \Delta \) on \( U \). Since \( \Gamma_0^k \times 0 \rightarrow \Gamma_0^k \times 1 \) on \( \Delta \), \( \tau (\Gamma_0^k \times 0) \sim \tau (\Gamma_0^k \times 1) \) on \( U \). But \( \tau (\Gamma_0^k \times 0) = \pi \pi_0 \Gamma_0^k \) and \( \tau (\Gamma_0^k \times 1) = \pi \pi_0 g \cdot \Gamma_0^k \). Since it is easily seen from the construction that this homology takes place on \((M \cdot H)^+\), the proof is complete.

Clearly the same proof suffices for the following lemma.

**Lemma 2.** Given any neighborhood \( M \) of \( e \) in \( G \), there is a neighborhood \( N \) of \( e \) such that for any closed subset \( X \) of \( H \) and any cycle \( \Gamma_{-k} \) of \( H \) mod \( X \), \( \Gamma_{-k} \sim g \cdot \Gamma_{-k} \) in \( M \cdot H \) mod \( M \cdot X \), whenever \( g \in N \).

**Lemma 3.** If \( D \) is an open connected subset of \( G \), then any two points
of $D$ lie on a compact connected subset of $D$.

**Proof.** Since $G$ is lc°, any neighborhood $O$ of a point $d$ of $D$ contains a neighborhood $W$ of $d$ with $W \subseteq D$ such that any point $w \in W$ lies, together with $d$, on a compact continuum in $O \cap D$. Now let $D_1$ be the set of all points of $D$ which can be joined to a fixed point $d_1 \in D$ by compact continua. Then, by the above, $D_1$ is both open and closed in $D$. Hence, since $D$ is connected, $D_1$ is all of $D$.

**Lemma 4.** If $O$ is a neighborhood of $e$ such that $C(O \cdot H)$ (where $C$ means closure) is not all of $G$, then $O \cdot H - H$ has at least two components.

**Proof.** Let $g$ be a point of $G - C(O \cdot H)$, and let $K$ be a compact connected set which contains both $e$ and $g$. Let $N$ be a neighborhood of $e$ in $O$, chosen by Lemma 1. A finite number of translations of $N$ cover $K$, and from these we may choose a sequence

$$e \in N, N_1, N_2, \ldots, N_k \ni g$$

where $N_i = g_i \cdot N$ and such that $N_i \cap N_{i+1} \neq \emptyset$. Let $\bar{g}_i$ be a point of $N_i \cap N_{i+1}$. Now, $\bar{g}_{i-1} \in g_i \cdot N$, so $g_i^{-1} \cdot \bar{g}_{i-1} \in N$. Hence, by Lemma 1,

$$\Gamma^{n-1} \sim g_i^{-1} \cdot \bar{g}_{i-1} \cdot \Gamma^{n-1}$$

where $\Gamma^{n-1}$ is a nonbounding cycle on $H$. Therefore,

$$g_i \cdot \Gamma^{n-1} \sim \bar{g}_{i-1} \cdot \Gamma^{n-1}.$$ 

Similarly, $\bar{g}_i \in g_i \cdot N$, and

$$g_i \cdot \Gamma^{n-1} \sim \bar{g}_i \cdot \Gamma^{n-1}.$$ 

Thus, we have

$$\Gamma^{n-1} \sim g \cdot \Gamma^{n-1}.$$ 

Now $\Gamma^{n-1} - g \cdot \Gamma^{n-1}$ is a cycle of $H \cup (G - O \cdot H)$ and $\Gamma^{n-1} - g \cdot \Gamma^{n-1} \sim 0$ in $G$. Hence [3, p. 227 (14.2)] there is in $G$ a cycle $\Gamma^n \bmod (H \cup (G - O \cdot H))$ such that $FT^n = \Gamma^{n-1} - g \cdot \Gamma^{n-1}$. Let $\bar{T}^n$ be the fundamental $n$-cycle of $G$, and let $T^n = \bar{T}^n$ on $G - (O \cdot H - H)$. Let $T^n = \bar{T}^n - T^1$.

In a neighborhood of any point of $O \cdot H - H$, $\Gamma^n$ is homologous to some multiple of $T^n$. If we assume that $O \cdot H - H$ is connected, then (cf. [1, p. 569]) this multiple, is the same for all points of $O \cdot H - H$, that is, $r \Gamma^n = T^n$. 

By definition, $T^1$ is on $H \cup (G - O \cdot H)$. Since $H$ and $G - O \cdot H$ are closed and disjoint, and since $\dim H < n$, $T^1$ must be on $G - O \cdot H$, so $FT^1$ is also on $G - O \cdot H$. 
But, from $0 = F\Gamma^n = F(T_1^n + T_2^n)$, we have
\[
F\Gamma_1^n = - F(T_2^n) = - F(r\Gamma^n) = - r(\Gamma^{n-1} - g\cdot \Gamma^{n-1}).
\]
This is not on $G - O \cdot H$, since $\Gamma^{n-1}$ is on $H$. Thus, the assumption that $O \cdot H - H$ has only one component leads to a contradiction.

We now choose a fixed connected neighborhood of $e$, satisfying the condition of Lemma 4, and denote by $J$ the product of $H$ by this neighborhood. We note that $J$ is a connected generalized $n$-manifold. It is not a group, but for any two elements of $J$ which are close enough to $H$, their product in $G$ is in $J$.

**Lemma 5.** $H$ is the boundary of each domain of $J - H$.

**Proof.** Let $D$ be any component of $J - H$. Since $J$ is $lc^0$, $D$ is open. Some point $h \in H$ is a limit point of $D$, or else $J$ would not be connected. Let $O$ be a neighborhood of $e$. Then $h \cdot O$ contains a point $d \in D$, that is, $h \cdot o = d$. Now $h^{-1} \cdot d = o \in O$. But $h^{-1} \cdot d$ is also in $D$. For $o \in H$, and, since $H$ is connected, $H \cdot o$ lies in one component of $J - H$. Since $h \cdot o = d$ is in $D$, $H \cdot o$ lies in $D$, and consequently, $e \cdot o = o$ is in $D$. Therefore $e$ is a limit point of $D$. Similarly, if $h$ is any other point of $H$, then the neighborhood $h \cdot O$ contains $h \cdot o$ which is in $H \cdot o$ and therefore in $D$. Thus, $h$ is a limit point of $D$, which proves the lemma.

**Lemma 6.** $J - H$ has just two components.

**Proof.** By Lemma 2, it is enough to show that $H$ does not have three complementary domains. Suppose there were three, $D_0, D_1$ and $D_2$. Let $p_1, p_2$ be points in $D_1$ and $D_2$, respectively, and let $Y_1, Y_2$ be neighborhoods of $p_1, p_2$ such that $\overline{Y}_i$ is compact and is in $D_i$.

$\gamma^0 = p_1 - p_2$ is a compact 0-dimensional cycle in $Y = Y_1 \cup Y_2$. $\gamma^0$ not $\sim 0$ in $J - H$, since $p_1$ and $p_2$ are in different components. But for any point $d_0 \in D_0, \gamma^0 \sim 0$ in $J - d_0 \cdot H$. For let $O$ be a neighborhood of $e$ not meeting $d_0 \cdot H$, which is in $D_0$, and let $O'$ be chosen so that every compact 0-cycle in $O'$ bounds in $O$. Choose $d_1 \in \partial O \cap D_1$ and $d_2 \in \partial O' \cap D_2$. Then $d_1 \sim d_2$ in $O$. By Lemma 3, $p_1 \sim d_1$ in $D_1, p_2 \sim d_2$ in $D_2$. Hence, $p_1 \sim p_2$ in $D_1 \cup D_2 \cup O$, which does not meet $d_0 \cdot H$.

Now, by Lemma 5.2 of [2], there is a compact cocycle $\gamma_n$ in $Y$ such that $(\Gamma^n \cdot \gamma_n) \sim \gamma^0$ in $J - H$, where $\Gamma^n$ is the fundamental $n$-cycle of $J$, and such that $\gamma_n \sim 0$ in $J - d_0 \cdot H$, for any $d_0 \in D_0$. Since $\gamma_n$ is a compact cocycle of $D_1 \cup D_2$, there is an infinite $n$-cycle $\Gamma_n$ of $D_1 \cup D_2$.

---

The main outline of this proof, and to some extent that of Lemma 7, is derived from Wilder [6, 7].
such that $KI(\Gamma^n, \gamma_n) = 1$. Let $\Gamma^{n-1} = FT^n$, so that $\Gamma^{n-1}$ is an infinite cycle of $H$.

We now choose a neighborhood $M$ of $e$ which does not meet $\overline{V}$, and a neighborhood $N$ satisfying the conditions of Lemma 1. Let $d_0 \subseteq D_0 \cap N$. Then $\Gamma^{n-1} \sim d_0 \cdot \Gamma^{n-1}$ in $M \cdot H$. Let $\Gamma^n = \{ \Gamma^n_i \}$ and let the chains involved in the homology $\Gamma^{n-1} \sim d_0 \cdot \Gamma^{n-1}$ be $\{ C^n_i \}$. Then $\{ \Gamma^n_i \} = \{ \Gamma^n_i - C^n_i \}$ is such that $FT_i^{n-1} = d_0 \cdot \Gamma^{n-1}_i$. By construction, $KI(\Gamma^n_i, \gamma^n_i) = KI(\Gamma^n_i, \gamma^n_0)$, since none of the chains $C^n_i$ meet $\overline{V}$.

$\{ \Gamma^n_i \}$ is not necessarily a Čech cycle. But, for each covering $U_\delta$, let $U_\rho(t)$ be an essential refinement (see [3, II 27: 13]) of $U_\delta$ relative to cycles of $J \cdot H$. Then $\{ \Gamma^n_i \} = \{ \pi_\rho(t) \Gamma^n_i \}$ is a Čech cycle mod $(d_0 \cdot H)^k$ and $KI(\Gamma^n_i, \gamma^n_0) = KI(\Gamma^n_i, \gamma^n_0)$ for all $\xi$.

But now we have reached a contradiction. For $\gamma_n \sim 0$ in $J - d_0 \cdot H$, so its Kronecker index with any infinite $n$-cycle of $J - d_0 \cdot H$ must be zero. But $KI(\Gamma^n_i, \gamma_n) = KI(\Gamma^n_i, \gamma_n) = 1$.

**Lemma 7.** For each point $h \in H$, $r^k(J - H, h) = 0$ for $1 \leq k \leq n - 1$ and $r^0(J - H, h) = 1$.

**Proof.** It is sufficient to consider the case $h = e$. Given any neighborhood $V$ of $e$, choose a neighborhood $V_1$ such that $C(\overline{C} \cdot (V_1 \cdot V_1))$ does not meet $\overline{V}_1$, where $B(V)$ is the boundary of $V$. Next choose a neighborhood $V_2$ such that if $\gamma^0 \subset V_2$, then $\gamma^0 \sim 0$ in $V_2$. Let $V_3$ be such that $\overline{V}_3 \cdot H$ does not contain all of $V_3$, and, finally, let $W$ be such that if $\gamma^k \subset W$, then $\gamma^k \sim 0$ in $V_3$. We assert that for $k \geq 1$, any $\gamma^k$ in $W - H$ bounds in $V - H$.

For let $A$ and $B$ be the two components of $J - H$ and let $\gamma^k = \gamma^k_A + \gamma^k_B$, where $\gamma^k_A$ is the part of $\gamma^k$ in $A$. Since $k \geq 1$, $\gamma^k_A$ is a cycle and it is sufficient to show that $\gamma^k_A \sim 0$ in $V \cap A$. If it does not, let $O$ be an open set in $W \cap A$ such that $\gamma^k$ is in $O$ and $O$ does not meet $H$. Then, by Lemma 5.2 of [2], there is a compact cocycle $\gamma_{n-k}$ in $O$ such that $\Gamma^n \cdot \gamma_{n-k} \sim \gamma^k$ in $O$, $\gamma_{n-k}$ not $\sim 0$ in $V \cap A$, and $\gamma_{n-k} \sim 0$ in $V_3$. Let $\Gamma^n$ be an infinite cycle of $V \cap A$ such that $KI(\Gamma^n, \gamma_{n-k}) = 1$.

In order to apply an argument similar to that of the preceding lemma, we choose a point of $B$ in the following fashion. Let $c$ be a point of $B$ in $V_2$ and not in $\overline{C} \cdot (V_2 \cdot H)$. By the choice of $V_2$, there is a continuum $K$ in $V_2$ which contains both $c$ and $e$. Let $M$ be a neighborhood of $e$ such that $M \cdot H$ does not meet $\overline{O}$. Choose $N$ by Lemma 1 and so that $N \cdot K$ is in $V_1$. $N \cdot K - H$ is an open subset of $J$ and hence is locally connected. Consequently, each component of $N \cdot K - H$ is also open. Let $C$ be that component which contains $c$. Since $C$ is open and $N \cdot K$ is connected, some point $h$ in $H$ must be a limit
point of \( C \). \( h \) is in \( V_1 \), since \( N \cdot k \) is in \( V_1 \), and therefore \( D = C \cdot h^{-1} \), containing \( d = ch^{-1} \), is an open connected subset of \( B \) and \( e \) is a limit point of \( D \). Also, since \( c \) is not in \( V_3 \cdot H \), neither is \( d \).

From \( N \cdot D \) and \( N \) itself a simple chain of regions running from \( e \) to \( d \) can be extracted, each element of the chain being a translation of \( N \).

Returning now to \( \Gamma^{n-k-1} \), let \( \Gamma^{n-k-1} \) be the part of \( F \Gamma^{n-k} \) on \( H \), so that \( \Gamma^{n-k-1} \) is a cycle of \( H \) mod \( X \), where \( X = \overline{V} \cap (H - V) \). Using the simple chain above, we have \( \Gamma^{n-k-1} \sim d \cdot \Gamma^{n-k-1} \) in \( M \cdot D \). Let the chains of this homology be \( \{ C^n_{r_k} \} \). Then \( \Gamma^{n-k}_{r_k} - C^{n-k}_{r_k} \), for each \( r_k \), is, by the choice of \( V_1 \), an infinite cycle of \( V \). Also, by the choice of \( M \), no \( C^{n-k}_{r_k} \) meets \( \overline{O} \), so \( KI((\Gamma^{n-k}_{r_k} - C^{n-k}_{r_k}) \cdot \gamma^{n-k}_{r_k}) = KI(\Gamma^{n-k}_{r_k} \cdot \gamma^{n-k}_{r_k}) = 1 \) for each \( r_k \).

Now we can proceed to the same contradiction we reached in the previous lemma, since \( \gamma^{n-k}_{r_k} - 0 \) in \( V_3 \) so its Kronecker index with any infinite cycle of \( V_3 \) is zero. This disposes of the case \( k \geq 1 \).

For \( k = 0 \), let \( \overline{\gamma}^0 \) be based on a pair of points, one in \( W \cap A \) and the other in \( W \cap B \). The proof used above applies to show that any \( \gamma^0 \) in \( W - H \) is homologous in \( V - H \) to a multiple of \( \overline{\gamma}^0 \).

**Lemma 8.** For each point \( h \) of \( H \), \( r_k(h) = 0 \) for \( k < n - 1 \) and \( r_{n-1}(h) = 1 \).

This is an immediate consequence of Theorem 6.2 of [2] and Lemma 7.

**Lemma 9.** \( H \) is \( l \cdot c^{n-1} \).

**Proof.** Given a neighborhood \( V \) of \( e \), choose \( V_1 \) in \( V \) such that any \( \gamma^{k+1} \) on \( \overline{V}_1 \) bounds in \( V \). Choose \( W \subseteq V_1 \) by Lemma 7 so that any \( \gamma^k \) in \( A \cap W \) bounds in \( A \cap V_1 \) and similarly for \( B \). We assert that any \( \gamma^k \) in \( W \cap H \) bounds on \( \overline{V} \cap H \).

To show this it is enough to show that for any neighborhood \( O \) of \( e \), \( \gamma^k \sim 0 \) in \( (O \cdot H) \cap \overline{V} \). In turn, to prove this it is sufficient to show that given any such \( \gamma^k \) and \( O \), and given any covering \( U_3 \), then there is a refinement \( U_3 \) such that \( \pi_2^k \sim 0 \) in \( (O \cdot H) \cap \overline{V} \).

By Lemma 1 we can choose a point \( a \in A \cap O \) such that \( \gamma^k \sim a \cdot \gamma^k \) in \( O \cdot (W \cap H) \) and we can choose a similar point \( b \in B \cap O \). By the choice of \( W \), \( a \cdot \gamma^k \sim 0 \) in \( A \cap V_1 \), and similarly for \( b \cdot \gamma^k \). Thus, we have families of chains \( \{ C^k_{a,k} \} \) and \( \{ C^k_{b,k} \} \) in \( O \cdot (W \cap H) \), \( \{ D^k_{a,k} \} \) in \( A \cap V_1 \) and \( \{ D^k_{b,k} \} \) in \( B \cap V_1 \) such that

\[
\begin{align*}
FC^k_{a,k} &= a \cdot \gamma^k, & FC^k_{b,k} &= b \cdot \gamma^k, \\
FD^k_{a,k} &= a \cdot \gamma^k, & FD^k_{b,k} &= b \cdot \gamma^k.
\end{align*}
\]

Hence, for each \( \gamma^k \), \( D^k_{a,k} - C^k_{a,k} + C^k_{b,k} - D^k_{b,k} \) is a cycle \( \delta^k_{l+1} \) on \( U_3 \) in
There is a refinement \( U_2 \) of \( U \) such that \( \pi_2 \delta^{k+1} \) is the coordinate of a Čech cycle, \( \delta^{k+1} \) on \( \overline{V} \). By the choice of \( V_1 \), \( \delta^{k+1} \sim 0 \) in \( V \), so there is a chain \( E^{k+2} \) on \( U \) such that

\[
FE^{k+2} = \pi_2 \delta^{k+2}.
\]

Let \( E^{k+2} = E^{k+2}_a + E^{k+2}_b \), where \( E^{k+2}_b \) is the part of \( E^{k+2} \) on \( \overline{B} \). Now

\[
FE^{k+2}_a - \pi_2 D^{k+1} = -C^{k+1} + C^{k+1}_b - D^{k+1}_b - FE^{k+2}_b.
\]

The chain on the right-hand side is in \( O \cdot B \) while that on the left is on \( \overline{A} \). Hence, since \( \overline{A} \cap \overline{B} = H \), \( E^{k+1} = FE^{k+2}_a - \pi_2 D^{k+1} \) is in \( O \cdot H \) and, of course, in \( V \). But

\[
F(-E^{k+1}) - F(\pi_2^{k+1} D_{a,2}) = \pi_2^{k+1} \cdot \gamma_a.
\]

Hence, \( \pi_2^{k+1} \cdot \gamma_a \sim 0 \) in \( (O \cdot H) \cap V \). But \( a \cdot \gamma_a \sim \gamma_a \) in \( O \cdot (W \cap H) \), so \( \pi_2^{k+1} \cdot \gamma_a \sim 0 \) in \( (O \cdot H) \cap V \).

At this point, we have shown, by Lemmas 8 and 9, that \( H \) has the local properties of a generalized manifold. To complete the proof it only remains to show that \( H \) is orientable, that is, that it carries an \((n-1)\)-cycle which is not carried by any proper closed subset of \( H \).

By Lemma 8, there are neighborhoods \( O_1 \) and \( O_2 \) of \( e \) such that there is an \((n-1)\)-cycle mod \( H - O_1 \) which does not bound mod \( H - O_2 \). By group translation, every point of \( H \) has associated with it such a non-bounding relative \((n-1)\)-cycle. Now an argument due to Smith [5] shows that we can carry through in the present situation the proof of Theorem 7.1 of [1] to obtain the desired \((n-1)\)-cycle.

In conclusion, we point out that by restricting \( G \), we can lighten the hypothesis on \( H \).

**Theorem.** Let \( G \) be a locally compact separable metric topological group which is also an orientable \( n \)-dimensional generalized manifold. Let \( H \) be a closed connected \((n-1)\)-dimensional subgroup. Then \( H \) is an orientable generalized manifold if any one of the following conditions is satisfied:

1. \( H \) separates some open set of \( G \).
2. For some open set \( O \) of \( H \), there is a nonbounding \((n-1)\)-cycle of \( H \) mod \( H - O \).
3. \( G \) is locally euclidean.

The Pontrjagin duality theorem for case (3) and Theorem 6.5 of [2] for case (2) show that both (3) and (2) imply (1). Now the proof of Lemma 1 of [4] shows that (1) yields a neighborhood of \( H \) which
is separated by $H$, that is, our Lemma 4. Since this is the only place in our proof where the original hypothesis on $H$ is used, the rest of the proof can remain unchanged.

In case (3), if $\dim G = 3$, we have Montgomery's theorem, for any 2-dimensional generalized manifold is locally euclidean [8].

**BIBLIOGRAPHY**


**Yale University**