RECURSION AND DOUBLE RECURSION

RAPHAEL M. ROBINSON

1. Introduction. We shall apply the results of PRF\(^1\) to construct by double recursion two functions which are not themselves primitive recursive, but which are related in interesting ways to the class of primitive recursive functions. In a sense, this note is a revised version of a paper by Rózsa Péter,\(^2\) much simplified by the use of PRF.

Let \(Sx\) denote the successor of \(x\). We shall say that a function \(G_nx\) of two variables \(n\) and \(x\) is defined by a double recursion from certain given functions, if

1. \(G_0x\) is a given function of \(x\).
2. \(G_{Sn}0\) is obtained by substitution from \(G_nz\) (considered as a function of \(z\)) and from given functions.
3. \(G_{Sn}Sx\) is obtained by substitution from the number \(G_{Sn}x\), from \(G_nz\) (considered as a function of \(z\)), and from given functions.

It is clear that if the given functions are primitive recursive, then \(G_nx\) is a primitive recursive function of \(x\) for each fixed \(n\). However, as we shall see, \(G_nx\) need not be a primitive recursive function of \(n\) and \(x\).

In §2, we shall show that the double recursion

\[
G_0x = Sx, \quad G_{Sn}0 = G_n1, \quad G_{Sn}Sx = G_nG_{Sn}x
\]

defines a function \(G_nx\) which majorizes all primitive recursive functions of one variable in the following sense: If \(Fx\) is a primitive recursive function of \(x\), then there exists a number \(n\) such that

\[
Fx < G_nx
\]

for all \(x\). It is also shown that \(G_nx\) is an increasing function of \(n\), so that

\[
Fx < G_xx
\]

for all sufficiently large \(x\). It follows that \(G_xx\) is not a primitive recursive function of \(x\), and hence that \(G_nx\) is not a primitive recursive

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function of \( n \) and \( x \). The above example is essentially the same as that given by Péter, which was a simplification of one previously given by Ackermann.\(^3\)

In §3, we shall determine two primitive recursive functions \( A(x) \) and \( B(x, y) \), such that the double recursion

\[
G_0 x = A x, \quad G_{S_n} 0 = 0, \quad G_{S_n} S x = G_n B(x, G_{S_n} x)
\]
defines a function \( G_x \) which generates all primitive recursive functions of one variable in the following sense: A primitive recursive function \( H(n, x) \) can be found, such that if \( F_x \) is a primitive recursive function of \( x \), then there exists a number \( n \) such that

\[
F_x = G_n H(n, x).
\]

It follows that \( G_x H(x, x) \) is not a primitive recursive function of \( x \), and hence that \( G_x x \) is not a primitive recursive function of \( n \) and \( x \). The above double recursion is of a much simpler form than the one given by Péter for a similar purpose. (In a later paper,\(^4\) she showed how all double recursions can be reduced to a standard form, which is however still not as simple as the above.) Also, the functions \( A(x) \) and \( B(x, y) \) which we use are comparatively simple; they can be obtained by substitution from constant and identity functions, and

\[
x + y, \quad x - y, \quad x^2, \quad [x^{1/2}], \quad [x/2], \quad [x/3].
\]

Here \( x - y = x - y \) if \( x \geq y \) and \( x - y = 0 \) otherwise. The function \( H(n, x) \) which we use is a certain quartic polynomial in \( n \) and \( x \).

Both of these results may be derived from PRF, §7, Theorem 3, which states that all primitive recursive functions of one variable can be obtained by starting with the two functions \( S \) and \( E \), and repeatedly using any of the formulas

\[
F_x = A x + B x, \quad F_x = B A x, \quad F_x = B = 0
\]
to construct a new function \( F \) from known functions \( A \) and \( B \). A second form of the result has \( E \) replaced by \( Q \), and \( A x + B x \) by \( |A x - B x| \). Here \( E x = x - [x^{1/2}]^2 \) is the excess of \( x \) over a square, and \( Q x = 0^E x \) is the characteristic function of squares.

2. The majorizing function. Let the function \( G_n x \) be defined by the double recursion

\[
G_0 x = S x, \quad G_{S_n} 0 = G_n 1, \quad G_{S_n} S x = G_n G_{S_n} x.
\]

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It is clear that $G_nx$ is never zero. We shall show that

$$G_nSx > G_nx,$$

so that for any $n$, $G_nx$ is a strictly increasing function of $x$, hence $G_nx > x$, that is, $G_nx \geq Sx$. In the first place, this is true for $n = 0$. Now assume for any value of $n$ and prove for $Sn$. We have indeed

$$G_{Sn}Sx = G_nG_{Sn}x > G_{Sn}x.$$

We shall show next that

$$G_{Sn}x \geq G_nSx.$$

This may be shown for a fixed $n$ by induction in $x$. For $x = 0$ we have equality by definition. Now assume the inequality for some value of $x$ and prove for $Sx$. By the inductive hypothesis and the inequality $G_nSx \geq SSx$, we have

$$G_{Sn}Sx = G_nG_{Sn}x \geq G_nG_nSx \geq G_nSSx,$$

as was to be shown. In particular, we have

$$G_{Sn}x > G_nx,$$

so that $G_nx$ is a strictly increasing function of $n$ for a fixed $x$.

The arguments used up to this point are the same as those given by Péter, although we have modified her function slightly. We shall now show that to every primitive recursive function $Fx$ there exists a number $n$ such that

$$Fx < G_nx.$$

We shall use the result quoted in §1 from PRF in the second form.

We must show that the conclusion holds for $Sx$ and $Qx$, and that if it holds for $Ax$ and $Bx$, then it also holds for $|Ax - Bx|$, $BAx$, and $B*0$. We have in the first place

$$Qx \leq Sx = G_0x < G_1x.$$

Now suppose that

$$Ax < G_kx, \quad Bx < G_1x.$$

If we set $n = \max (k, l)$, then

$$Ax < G_nx, \quad Bx < G_nx.$$

Hence

$$|Ax - Bx| < G_nx, \quad B*0 < G_0G_n1 = G_{Sn}x,$$
and
\[ BA x < G_n G x < G_n G s_n x = G s_n S x \leq G s s_n x. \]

3. The generating function. We shall make use of pairing functions \( J(u, v), Kx, Lx \), that is, functions which satisfy
\[ KJ(u, v) = u, \quad LJ(u, v) = v. \]

Such functions establish a one-to-one correspondence between all pairs of numbers and some numbers. We shall want to use functions \( J(u, v), Kx, Lx \), which are primitive recursive, and the conditions \( Kx \leq x, Lx \leq x \) will be needed. We shall also suppose, as in PRF, §4, that \( J(0, 0) = 0 \), and that if \( Ls x > 0 \), then \( ks x = Kx \) and \( Ls x = sLx \); the interpretation of these conditions is discussed there. Suitable functions are
\[ J(u, v) = ((u + v)^2 + u)^2 + v, \quad Kx = E[x^{1/2}], \quad Lx = Ex. \]

Since for such pairing functions, \( Kx = u \) and \( Lx = v \) have infinitely many solutions for \( x \) when \( u \) and \( v \) are given, we can extend the correspondence from one between pairs of numbers and numbers to one between triples of numbers and numbers, by finding two primitive recursive functions \( J(u, v, w) \) and \( Mx \), with \( J(u, v, 0) = J(u, v) \), and such that
\[ KJ(u, v, w) = u, \quad LJ(u, v, w) = v, \quad MJ(u, v, w) = w. \]

Since \( J(0, 0, 0) = 0 \), we see that \( K0 = 0, L0 = 0, M0 = 0 \). Suitable functions extending the \( J(u, v), Kx, Lx \) given above are
\[ J(u, v, w) = ((u + v + w)^2 + u)^2 + v, \quad Mx = \left[ x^{1/4} \right] - (Kx + Lx). \]

We now define \( F_n x \) by the formulas
\[ F_0 x = Ex, \quad F_1 x = Sx, \]
\[ F_{3u+2} x = F_{Ku} x + F_{Lu} x, \quad F_{3u+3} x = F_{Lu} F_{Ku} x, \quad F_{3u+4} x = F_{u}^2 0. \]

According to the theorem of PRF quoted in §1, if \( Fx \) is a primitive recursive function, then there exists a number \( n \) such that \( Fx = F_n x \).

We shall now define a function \( Gx \) (which is not primitive recursive) such that \( G(2v) = F_{Kv} Lv + Mv \). The definition of \( Gx \) is completed by supposing that for \( v > 0 \)
\[ G(2v - 1) = \begin{cases} \frac{F_{Kv} Lv}{2} & \text{if } Kv = 3u + 2 \text{ or } 3u + 3, \\ \frac{F_{Kv} PLv}{2} & \text{if } Kv = 3u + 4, Lv > 0, \\ 0 & \text{otherwise}. \end{cases} \]
Here $P_y = y-1$ is the predecessor of $y$.

The definition of $G(2v)$ was so chosen that

$$F_{n,x} = G(2J(n, x)).$$

Thus $Gx$ may be considered as generating all primitive recursive functions. The essential numbers $F_{K,v}L_v$ were put in alternate places and modified by adding $Mv$, and then such values were interpolated for $G(2v-1)$ that $Gx$ would satisfy a functional equation

$$GSx = GB(x, Gx),$$

where $B(x, y)$ is a primitive recursive function. Indeed, we shall define $B(x, y)$ so that the function determined from suitable initial values by the formula $G_{Sn}Sx = G_nB(x, G_{Sn}x)$ will approximate to $Gx$ in such a way that

$$F_{n,x} = G_n(2J(n, x)).$$

Thus all primitive recursive functions will be generated by a double recursion.

The definition of $B(x, y)$ is the following: For every $v > 0$, let

$$B(2v - 2, y) = \begin{cases} 2J(Ku, L_v, 0) & \text{if } K_v = 3u + 2 \text{ or } 3u + 3, \\ 2J(0, 0, y + Mv) & \text{if } K_v = 3u + 4, L_v > 0, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$B(2v - 1, y) = \begin{cases} 2v & \text{if } K_v = 0 \text{ or } 1, \\ 2J(L_u, L_v, y + Mv) & \text{if } K_v = 3u + 2, \\ 2J(L_u, y, Mv) & \text{if } K_v = 3u + 3, \\ 2J(0, 0, Mv) & \text{if } K_v = 3u + 4, L_v = 0, \\ 2J(u, y, Mv) & \text{if } K_v = 3u + 4, L_v > 0. \end{cases}$$

We proceed to verify that $GSx = GB(x, Gx)$ in all cases.

$x = 2v - 2, K_v = 3u + 2$ or $3u + 3$:

$$GB(x, Gx) = F_{K_u}L_v = GSx.$$

$x = 2v - 2, K_v = 3u + 4, L_v > 0$:

$$GB(x, Gx) = F_0 + (Gx + Mv) = F_{K_v}L_Pv = F_{K_v}PL_v = GSx.$$

$x = 2v - 2$, other cases:

$$GB(x, Gx) = G0 = 0 = GSx.$$
Let the function $A_x$ be defined by

$$A_x(2v) = \begin{cases} 
ELv + Mv & \text{if } Kv = 0, \\
SLv + Mv & \text{if } Kv = 1, 
\end{cases}$$

and $A_x = 0$ otherwise. Then the double recursion $G_s x = A_x$, $G_{s_n} 0 = 0$, $G_{s_n} x = G_n B(x)$ defines a function $G_n x$, which we shall show approaches $G x$ as $n$ increases in such a way that

$$G_n(2v) = G(2v) \quad \text{if } Kv \leq S_n,$$

and also for $v > 0$

$$G_n(2v - 1) = G(2v - 1) \quad \text{if } Kv \leq S_n.$$
It remains to consider the case \( 2 \leq K_v \leq SSn \). Now if \( K_v = 3u+2, 3u+3, \) or \( 3u+4 \), we see that for any \( y \), \( B(2v-2, y) \) and \( B(2v-1, y) \) are both of the form \( 2w \) with \( Kw \leq u < Kv \), hence \( Kw \leq Sn \). It follows that

\[
G_n B(2v - 2, y) = GB(2v - 2, y), \quad G_n B(2v - 1, y) = GB(2v - 1, y).
\]

Hence

\[
G_{Sn} Sx = G_n B(x, G_{Sn} x) = GB(x, G_{Sn} x) \quad \text{for } x = 2v - 2 \text{ or } 2v - 1.
\]

We shall now prove by induction in \( v \) that if \( 2 \leq K_v \leq SSn \) then

\[
G_{Sn} (2v - 1) = G(2v - 1), \quad G_{Sn} (2v) = G(2v),
\]

which will complete the proof. For a given value of \( v \), we shall derive both of these, assuming that

\[
G_{Sn} (2v - 2) = G(2v - 2)
\]

provided that \( 2 \leq KPv \leq SSn \). We have indeed

\[
G_{Sn} (2v - 1) = GB(2v - 2, G_{Sn} (2v - 2)) = GB(2v - 2, G(2v - 2)) = G(2v - 1),
\]

since \( B(2v - 2, y) \) depends on \( y \) only if \( K_v \) has the form \( 3u+4 \) and \( L_v > 0 \), in which case \( KPv = K_v \), so that the inductive hypothesis may be used. Finally,

\[
G_{Sn} (2v) = GB(2v - 1, G_{Sn} (2v - 1)) = GB(2v - 1, G(2v - 1)) = G(2v).
\]

Let

\[
H(n, x) = 2J(n, x).
\]

If \( Fx \) is any primitive recursive function of \( x \), then there exists an \( n \) such that \( Fx = F_n x \). From what we have proved, we see that

\[
Fx = F_n x = G(2J(n, x)) = G_n (2J(n, x)) = G_n H(n, x).
\]

thus establishing the result stated in the introduction.

University of California