the subject matter. Solutions of more difficult exercises are given or indicated in the text.

HING TONG

Integration in finite terms. By J. F. Ritt. New York, Columbia University Press, 1948. 7+100 pp. $2.50.

Advances in mathematical analysis have been intimately bound to the development of the notion of function. It was only in the last century that, both in the real and complex domain, the function concept was explicitly and completely elaborated. With this achievement it was possible to place on a sound foundation the calculus of earlier times. Having progressed so far, certain questions concerning the properties of functions of classical analysis lost their logical import. But their historical significance remains untarnished as generation after generation of young mathematicians is trained through the medium of the calculus.

The functions of classical analysis are the elementary functions: that is, those which can be constructed from the variables $x, y, \cdots$ by a finite number of algebraic operations and the taking of logarithms and exponentials. For example, $\cos y^{1/2}+\log [x^3+\arctan (x \log y)]$ is elementary. The young student quickly discovers that the closure of this set of functions under the operations of analysis is not an obvious fact, if it be a fact at all. Indeed, his instructor assures him that certain functions cannot be integrated in finite terms, that is, their integrals cannot be given an elementary representation. One hazards the guess that in a substantial number of the good courses in calculus offered in this and other countries, this is the one subject about which the instructor may not have first-hand knowledge. Rather, he imparts to his young charges, frequently with embarrassment, information which is based on hearsay.

Professor Ritt has now written a short book in which the reader will find all the material on this subject which should be the property of the complete mathematician. The theory exposed is one of considerable charm and of classical importance. A method is developed which may be used in attempting the solution of arbitrary problems on the representation of functions in an elementary manner. This method is then applied to certain specific questions where it yields complete results. It does not provide easy answers to the great variety of questions which one could propose. In spite of this fact, it possesses a certain degree of finality. It is truly extraordinary that no book has been written previously on this subject except in Russian.[1]

[1] The book by G. H. Hardy, *The integration of functions of a single variable*, Cambridge Tracts, 1905, is not one in which the Liouville theory is expounded. The author

One may well propose that no other work need be printed for the next fifty years since our knowledge will hardly alter essentially in that period.

The material is so arranged as to be accessible to a good graduate student. The reader must be equipped with sound knowledge of the theory of functions of a complex variable including the elements of analytic continuation, Riemann surfaces of algebraic functions, and power series expansions of algebraic functions. The exposition is endowed with great clarity. The author bases his style on the staccato sentence. This makes the climb steady, the footing secure.

The founder of the theory here considered was Joseph Liouville. Over one hundred years ago he classified the elementary functions and discovered the important principles with which questions concerning them can be resolved. He determined the form which the integral of an algebraic function must have if it is elementary. He showed that the elliptic integrals of the first and second kind are not elementary. He investigated the integral of $e^{g(x)} \cdot y(x)$ where $g(x)$ and $y(x)$ are algebraic; this leads to the proof of the fact that e^{x^2} does not have an elementary integral. Introducing quadratures as an allowable operation, he showed that the Riccati equation $y'+y^2=x^n$ can be integrated in *finite* terms only if (the "if" is due to Daniel Bernouilli) $n=-2$ or $n=-4p/(1+2p)$ with p an integer. Thus, the Bessel equation with parameter ν has a finite solution if and only if 2ν is an odd integer. All these results are to be found in this monograph.

Besides Liouville, the leading other contributors to this theory have been the author himself and the Russian mathematician, D. Mordukhai-Boltovskoi. Results of these men here presented are concerned with the solution of problems by elementary methods in *implicit* terms. For example, the author reproduces his proof that if $y(x)$ is elementary and $w(x)$ is an integral of $y(x)$ then the existence of an elementary relation of two variables, $F(x, w)=0$, implies that w is itself an elementary function of x. A recent contribution of Ostrowski is also included: The question of solving problems in finite terms is essentially based on the notion of field extension. One begins with a differential field $\mathcal{J}$. If $\mathcal{J}$ is not sufficiently large for our purposes, successive adjunctions of the exponentials or logarithms of functions in $\mathcal{J}$ yield a field extension $\mathcal{J}'$. In $\mathcal{J}'$ one may look for functions which satisfy our desiderata. With these ideas one is in a position to extend Liouville's result about the integral of an algebraic function to a theorem concerning the integral of a function algebraic over a given differential field $\mathcal{J}$.

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does briefly mention certain of Liouville's results. He gives no proofs but refers the reader to original sources.

A frequently recurring difficulty permeates phases of our undergraduate teaching. Even our better students are puzzled and annoyed by a situation which could easily be improved. We are at fault in that we do not formally classify the distinct domains (types of function or types of numbers) in which we operate. Thus we state successively that $1/\log x$ cannot be integrated; that the differential equation $y'-1/\log x=0$ can be solved whereas others more complicated cannot; finally that all differential equations can be solved by series. Likewise, in the equation $y+\log y=x$ it is impossible to solve for y; yet the equation defines y as a function of x and hence we may compute dy/dx with the help of standard theorems on differentiation. The worst paradox of all is that since $x^2+1=0$ has no root, we must use a special symbol to represent it. It would seem that our students would thrive better if we gave less attention to the introduction of ϵ and δ and devoted a little time to the discussion of these questions. One ventures the prediction that future texts in the calculus will carry out such a program explicitly. The publication of *Integration in finite terms* may accelerate this very desirable end by making the Liouville theory current coin in mathematical circles.

E. R. LORCH

Matrix and tensor calculus with applications to mechanics, elasticity and aeronautics. By A. D. Michal. (Galcit Aeronautical series.) New York, Wiley; London, Chapman and Hall, 1947. 13+132 pp. $3.00.

The author states in the preface that the purpose of his book is "to give the reader a working knowledge of matrix calculus and tensor calculus, which he may apply to his own field." To accomplish that much in 130 pages is a difficult problem. Professor Michal attempts to solve it by omitting many proofs and by restricting severely the material presented. Thus several basic notions (for example, characteristic vectors of a matrix) are not mentioned. These omissions are partly compensated for by numerous *Notes* collected at the end of the volume and by an extensive bibliography. On the other hand, this book contains information on some non-standard topics. These include the theory of "multiple-point tensor fields" (originated by the author two decades ago), and a tensor treatment of the boundary layer theory (due to Lin).

The book consists of two largely independent parts, one dealing with matrices, the other with tensors. Each part begins with the fundamental definitions and theorems. Further mathematical concepts are introduced in connection with concrete applications which range over various fields of mechanics. While no problem is pursued