ON THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER INVARIANT UNDER CONTACT TRANSFORMATIONS

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In the classical Lie theory it is shown how to construct a differential equation invariant under a given group, and how to solve an equation when a group leaving the equation invariant is known. However, little is said about the problem of determining the group for a given differential equation, which is by far the most interesting problem.

In the present paper, necessary and sufficient conditions for the existence of an infinitesimal contact transformation leaving a given equation invariant are determined along with the general form of the characteristic function of the group. It will also be shown how to reduce, by a proper change of variables, the infinitesimal contact transformation to a point transformation. This enables one to solve the transformed differential equation by Lie’s methods. Passing back to the original variables, a new differential equation is obtained which combined with the original equation gives its solution in parametric form.

Let

\[ Bf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \pi \frac{\partial f}{\partial p} \]

be the symbol of the infinitesimal contact transformation leaving invariant the differential equation \( u = F(v) \), with \( u = u(x, y, p) \), \( v = v(x, y, p) \), \( p = dy/dx \), and \( F \) such that the equation \( G(x, y, p) = u - F(v) = 0 \) satisfies the various conditions for the existence of solutions (but otherwise arbitrary). Throughout this paper we shall assume that:

(A) Both \( u \) and \( v \) have first derivatives with respect to \( x, y \) and \( p \), at least in some region \( R \) of the \((x, y, p)\)-space.

(B) The Jacobians

\[ J_1 = \frac{\partial(u, v)}{\partial(y, p)}, \quad J_2 = \frac{\partial(u, v)}{\partial(p, x)}, \quad J_3 = \frac{\partial(u, v)}{\partial(x, y)} \]

have in \( R \) derivatives of the first and second orders, while \( J_1 \) and \( J_2 \) have also derivatives of the third order with respect to \( x, y \) and \( p \),

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as are involved in the discussion.

(C) The functions \( u \) and \( v \) are not in involution, that is,

\[
[u,v] = \begin{vmatrix}
    u_x + pu_v & u_x + pu_v \\
    v_x + pv_y & v_x + pv_y
\end{vmatrix} = J_2 - pJ_1 \neq 0.
\]

Since \( u \) and \( v \) are to be invariants under \( Bf \) they will satisfy the partial differential equations

\[
\begin{align*}
\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \pi \frac{\partial u}{\partial p} &= 0, \\
\xi \frac{\partial v}{\partial x} + \eta \frac{\partial v}{\partial y} + \pi \frac{\partial v}{\partial p} &= 0,
\end{align*}
\]

from which we obtain

\[
\frac{\xi}{\partial(u, v)/\partial(y, p)} = \frac{\eta}{\partial(u, v)/\partial(p, x)} = \frac{\pi}{\partial(u, v)/\partial(x, y)} = \sigma,
\]

\( \sigma = \sigma(x, y, p) \) being the common ratio. This can be written

\[\text{(1)} \quad \xi = \sigma J_1, \quad \eta = \sigma J_2, \quad \pi = \sigma J_3.\]

If \( W \) is the so-called characteristic function of the infinitesimal contact transformation, we have also

\[\text{(2)} \quad W = p\xi - \eta = \sigma(pJ_1 - J_2).\]

Now to find \( \sigma \) we recall that\(^1\)

\[\text{(3)} \quad \xi = \frac{\partial W}{\partial p}, \quad \pi = -\frac{\partial W}{\partial x} - p \frac{\partial W}{\partial y}.\]

As a consequence of (1), (2) and (3) we obtain the system of equations

\[\begin{align*}
(pJ_1 - J_2) \frac{\partial \sigma}{\partial p} + \left( \frac{p}{p} \frac{\partial J_1}{\partial p} - \frac{\partial J_2}{\partial p} \right) \sigma &= 0, \\
(pJ_1 - J_2) \frac{\partial \sigma}{\partial x} + p(pJ_1 - J_2) \frac{\partial \sigma}{\partial y} + \left[ \left( \frac{p}{p} \frac{\partial J_1}{\partial x} - \frac{\partial J_2}{\partial x} \right) + p \left( \frac{p}{p} \frac{\partial J_1}{\partial y} - \frac{\partial J_2}{\partial y} \right) + J_3 \right] \sigma &= 0.
\end{align*}\]

This system may be written in the homogeneous form

\(^1\) See Cohen, \textit{An introduction to the Lie theory of one-parameter groups}, p. 186.
Differential Equations of the First Order

\[ A_1 f = \frac{\partial f}{\partial p} + M_1 \frac{\partial f}{\partial \sigma} = 0, \]

\[ A_2 f = \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial y} + M_2 \frac{\partial f}{\partial \sigma} = 0, \]

in which

\[ M_1 = -\sigma \frac{\rho (\partial J_1 / \partial p) - (\partial J_2 / \partial p)}{\rho J_1 - J_2}, \]

\[ M_2 = -\sigma \frac{(\rho (\partial J_1 / \partial x) - \partial J_2 / \partial x) + \rho (\partial J_1 / \partial y) - \partial J_2 / \partial y) + J_3}{\rho J_1 - J_2}. \]

Adjoining to the system (5) the equations

\[ A_3 f = (A_1 A_2) f = \frac{\partial f}{\partial y} + (A_1 M_2 - A_2 M_1) \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial y} + M_3 \frac{\partial f}{\partial \sigma} = 0, \]

\[ A_4 f = (A_1 A_3) f = (A_1 M_2 - A_3 M_1) \frac{\partial f}{\partial \sigma} = 0, \]

\[ A_5 f = (A_2 A_3) f = (A_2 M_2 - A_3 M_2) \frac{\partial f}{\partial \sigma} = 0, \]

we see that the equations

\[ A_1 M_3 - A_3 M_1 = 0, \quad A_2 M_3 - A_3 M_2 = 0, \]

are necessary and sufficient conditions in order that the system (4) have a solution. The system (5)–(8) implies the Jacobian complete system

\[ K_1 f = \frac{\partial f}{\partial x} + (M_2 - \rho M_3) \frac{\partial f}{\partial \sigma} = 0, \]

\[ K_2 f = \frac{\partial f}{\partial y} + M_2 \frac{\partial f}{\partial \sigma} = 0, \]

\[ K_3 f = \frac{\partial f}{\partial \rho} + M_1 \frac{\partial f}{\partial \sigma} = 0. \]

Either we may solve (12) or the equivalent total differential equation

\[ (M_2 - \rho M_3) dx + M_3 dy + M_1 dp - d\sigma = 0. \]

If \( f = \psi(x, y, p, \sigma) \) is the solution of (12), then
(14) \[
\psi(x, y, p, \sigma) = c
\]
will be the solution of (13), and conversely. Equation (14) determines \( \sigma \) in terms of \( x, y, \) and \( p. \) Since \( \sigma \) enters as a factor in \( M_1 \) and \( M_2, \) it is also a factor of \( M_3. \)

Hence, equation (13) can be written

\[
d\sigma/\sigma = d\omega(x, y, p),
\]

and so \( \sigma \) has the form

(15) \[
\sigma = ke^{u(x, y, p)}.
\]

Several special formulas for \( \sigma \) may be found. For instance, if

\[
M_1 = \phi_1(p)\sigma, \quad M_2 = \phi_2(x)\sigma,
\]

then \( M_3 = 0, \) and equation (13) reduces to

\[
\phi_2(x)\sigma dx + \phi_1(p)\sigma dp - d\sigma = 0,
\]

from which we obtain

\[
\sigma = k \exp \left( \int \phi_1(p) dp + \int \phi_2(x) dx \right).
\]

Therefore, the characteristic function takes the form

(16) \[
W = k(pJ_1 - J_2) \exp \left( \int \phi_1(p) dp + \int \phi_2(x) dx \right)
\]

by virtue of (2). This special case will be of use in some examples to be considered later.

We summarize our results in the following theorem.

**Theorem.** The characteristic function \( W \) of the infinitesimal contact transformation leaving invariant a given differential equation \( u = F(v) \) can be found by the formula

\[
W = k(pJ_1 - J_2) e^{u(x, y, p)}
\]

if, and only if, the equations

\[
A_1M_3 - A_2M_1 = 0, \quad A_2M_3 - A_3M_2 = 0
\]

are both satisfied for all values of \( x, y \) and \( p. \)

Now, to solve the differential equation \( u = F(v) \) invariant under the known group

\[
\text{If } M_1 = \sigma N_1, \quad M_3 = \sigma N_3, \quad \text{then } M_4 = A_1M_2 - A_2M_1 = \sigma(\partial N_2/\partial p - \partial N_1/\partial x - p\partial N_1/\partial y).
\]

This relation, together with (11), are the conditions in order that (13) be an exact differential when divided by \( \sigma. \)
\[ Bf = W_x \frac{\partial f}{\partial x} + (\varphi W_y - W) \frac{\partial f}{\partial y} - (W_x + \varphi W_y) \frac{\partial f}{\partial \varphi}, \]

we consider two cases:

(A) Both \( \xi = W_\varphi \) and \( \eta = \varphi W_\varphi - W \) are free of \( \varphi \). This case occurs when \( W \) is linear in \( \varphi \). Then \( Bf \) represents an extended point transformation and the equation may be solved by introducing canonical variables.

(B) Either \( \xi \) or \( \eta \), or both, contain \( \varphi \). Then \( Bf \) represents a general contact transformation.

In this case we may show that by a suitable change of variables the transformation reduces to a point transformation. To this aim, let us define a finite contact transformation

\[\begin{align*}
X &= X(x, y, \varphi), \\
Y &= Y(x, y, \varphi), \\
P &= P(x, y, \varphi)
\end{align*}\]

in the following manner: \( X = u, \ Y \neq X \) in involution with \( X \), that is, such that \( [XY] = 0 \), or

\[ X_x \frac{\partial Y}{\partial x} + \varphi X_y \frac{\partial Y}{\partial y} - (X_x + \varphi X_y) \frac{\partial Y}{\partial \varphi} = 0, \]

and \( P \) by the equation \( P = Y_x/X_x \). The symbol for the transformed group will be

\[ Bf = \frac{\partial f}{\partial X} + \frac{\partial f}{\partial Y} + \frac{\partial f}{\partial P} = BX \frac{\partial f}{\partial X} + BY \frac{\partial f}{\partial Y} + BP \frac{\partial f}{\partial P}. \]

But \( \xi = BX = Bu = 0 \) since \( u \) is invariant under \( Bf \). Since \( \xi = W_\varphi \) this implies that \( W \) is free of \( P \). Also, we find that \( \tilde{\eta} \) does not contain \( P \) because \( \tilde{\eta} = P W_\varphi - W = -W \). Hence, \( Bf \) is an extension of the point transformation group

\[ Uf = -\frac{\varphi}{W(X, Y)} \frac{\partial f}{\partial Y}. \]

This group can be reduced further by introducing the canonical variables

\[ X^* = X, \quad Y^* = -\int \frac{\partial Y}{W(X, Y)}. \]

Then the symbol of the infinitesimal transformation assumes the form

\footnote{Cohen, loc. cit. p. 195, proves that the contact transformation reduces to a point transformation by assuming the corresponding differential equation solvable for \( \varphi \) in the form \( \varphi = \omega(x, y) \).}
The equation \( u = F(v) \) when written with the variables \( X, Y, P \) takes the form

\[ (22) \quad \phi(X, Y, P) = 0. \]

This is also a differential equation, that is, \( P = dY/dX \), since the relation \( dY - PdX = \lambda(dy - pdx) \) which holds for any contact transformation implies \( dY - PdX = 0 \) whenever \( dy - pdx = 0 \). Since (22) will be invariant under (21) we are in position to solve (22), either directly or by introducing the canonical variables \( X^*, Y^* \) [this last step reduces the equation to the form \( dY^*/dX^* = G(X^*) \)]. Let

\[ (23) \quad \psi(X, Y, c) = 0 \]

be the solution of (22). Passing back to the original variables we get a second differential equation

\[ \Psi(x, y, p, c) = 0 \]

which together with \( u = F(v) \) determines the integral curves of the latter in terms of the parameter \( p \).

**Examples.** I. Consider the differential equation

\[ (24) \quad \frac{d}{dx} + \frac{y}{p} = F(x + 2p). \]

Here \( u = p + y/p, v = x + 2p \). Hence, it follows that \( J_1 = 2/p, J_2 = 1 - y/p^2, J_3 = -1/p, pJ_1 - J_2 = 1 + y/p^2, M_1 = 2\sigma/p, M_2 = M_3 = 0. \)

Formula (16) can be applied with \( \phi_1(p) = 2/p, \phi_2(x) = 0. \) Therefore, the characteristic function of the group is

\[ W = k(1 + y/p^2)p^2 = k(p^2 + y). \]

Since a constant factor is irrelevant, we see that equation (24) is invariant under the infinitesimal contact transformation

\[ Bf = 2p \frac{\partial f}{\partial x} + (p^2 - y) \frac{\partial f}{\partial y} - p \frac{\partial f}{\partial p}. \]

By taking \( X = v = x + 2p \) equation (19) reduces to

\[ (25) \quad 2 \frac{\partial Y}{\partial x} + 2p \frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial p} = 0. \]

The corresponding system of ordinary differential equations is
\[
\frac{dx}{2} = \frac{dy}{2\rho} = \frac{d\rho}{-1} = \frac{dY}{0},
\]
from which we obtain \( Y = \rho^2 + y \) as a particular integral of (25). Finally we have \( \rho = 2\rho/2 = \rho \). Introducing the new variables in (24) we get \( dY/Y = dX/F(X) \). Hence, we have

\[
Y = ce^{\alpha(x)}, \quad G(X) = \int \frac{dX}{F(X)}.
\]

Passing back to the variables \( x, y, \rho \) we obtain

\[
(26) \quad \rho^2 + y - ce^{\alpha(x+y)} = 0.
\]

The system (24)-(26) furnishes the solution of the equation (24). For instance, if \( F(x+2\rho) = \tan (x+2\rho) \), equations (24) and (26) are respectively

\[
\rho + y/\rho = \tan (x + 2\rho), \quad \rho^2 + y = c \sin (x + 2\rho).
\]

Solving for \( x \) and \( y \) we find

\[
x = -2t + \arccos (t/c),
\]
\[
y = -t^2 \pm (c^2 - t^2)^{1/2},
\]
which are the parametric equations of the solution, where \( t = \rho \) is the parameter.

II. To apply the method to find the group leaving invariant some familiar types of ordinary differential equations, let us consider first the homogeneous equation

\[
\frac{d\rho}{\rho} = F(y/x).
\]
We have \( u = \rho, \ v = y/x, \ J_1 = -1/x, \ J_2 = -y/x^2, \ J_3 = 0, \ \rho J_1 - J_2 = (y - px)/x^2, \ M_1 = 0, \ M_2 = 2\sigma/x, \ M_3 = 0 \). By using formula (16) with \( \phi_1(\rho) = 0, \ \phi_2(x) = 2/x \), we get (taking \( k = -1 \))

\[
W = \rho x - y.
\]

Since \( W \) is linear in \( \rho \) we obtain the point transformation with symbol

\[
Uf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y},
\]
which corresponds to the so-called homotetic transformation.

For the linear equation \( \rho + P(x)y = F(x) \) we have \( u = \rho + Py, \ v = x, \ J_1 = 0, \ J_2 = 1, \ J_3 = -P, \ \rho J_1 - J_2 = -1, \ M_1 = 0, \ M_2 = -\sigma P, M_3 = 0 \).
By putting $\phi_1(\rho) = 0, \phi_2(x) = -P, ke^\varepsilon = 1$ in formula (16) we obtain

$$W = -\exp \left( - \int Pdx \right).$$

Hence the symbol for the group has the form

$$Uf = \exp \left( - \int Pdx \right) \frac{df}{dy}.$$

III. Finally, we shall give a short table of some general types with the corresponding characteristic functions.\footnote{I am indebted to my former students Miss C. Santana and Dr. R. Peña for the fourth and fifth types shown in the list.}

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<td>$y = \rho x + F[x\phi(\rho)]$</td>
<td>$k\rho \phi(\rho)$</td>
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</tr>
<tr>
<td>$y + \phi(\rho) = \rho F\left[x + \int \phi'(\rho)d\rho/\rho\right]$</td>
<td>$k[y + \phi(\rho)]$</td>
</tr>
<tr>
<td>$e^\varepsilon \phi(x + y + \rho) = F[e^\varepsilon(\rho + 1)]$</td>
<td>$k\phi(x + y + \rho)$</td>
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<tr>
<td>$\frac{y + x\phi(\rho)}{\rho + \phi(\rho)} = F\left[\log x + \int \frac{\phi'(\rho)d\rho}{\rho + \phi(\rho)}\right]$</td>
<td>$k[y + x\phi(\rho)]$.</td>
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