A NOTE ON THE ERGODIC THEOREMS

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Introduction, definitions and remarks. The purpose of this note is to give an example of a measurable transformation of a measure space onto itself for which the individual ergodic theorem holds while the mean ergodic theorem does not hold.

Let $S$ be a measure space of finite measure, $m$ the measure defined on the measurable subsets of $S$, and $T$ a 1-1 point transformation of $S$ onto itself which is measurable (both $T$ and $T^{-1}$ transform measurable sets into measurable sets). Let the points of $S$ be denoted by $y$ and let $f(y)$ be any real valued function defined on $S$. We denote by $F_h(y)$ the average $(1/h) \sum_{i=0}^{h-1} f(T^i y)$.

We shall say that the individual ergodic theorem holds for $f(y)$ if the sequence of averages $\{F_h(y)\}$ converges to a finite limit almost everywhere. If the individual ergodic theorem holds for every integrable function $f \in L_1(m)$ we shall say that the individual ergodic theorem holds (with respect to $m$).\(^1\)

We shall say that the mean ergodic theorem holds in $L_p(m)$ ($p \geq 1$) for a function $f \in L_p(m)$ if $F_h(y) \in L_p(m)$ for $h=1, 2, \cdots$ and the sequence $\{F_h(y)\}$ converges in the norm of $L_p(m)$. If the mean ergodic theorem holds in $L_p(m)$ for every function $f(y) \in L_p(m)$ then we shall say that the mean ergodic theorem holds in $L_p(m)$.

The following relations between the two ergodic theorems are known: If $T$ is measure preserving, both the individual [1] and the mean [4]\(^2\) ergodic theorems hold. Without assuming that $T$ is measure preserving, the mean ergodic theorem in $L_p(m)$ for any $p \geq 1$ implies the individual ergodic theorem for all functions in $L_p(m)$ ([2, p. 1061], see also [3, p. 539] for the case $p = 1$).

The question arises whether, conversely, the individual ergodic theorem implies the mean ergodic theorem in $L_p(m)$ for some $p \geq 1$. This question has significance only when $L_p(m)$ is transformed into itself by the transformation induced on it by $T$. For in this case and only in this case is it true that for any $f \in L_p(m)$ the averages $\{F_h\}$ also belong to $L_p(m)$ for $h=1, 2, 3, \cdots$\(^3\). We answer this question in

\(^1\) The words in the parenthesis will be omitted if there is no reason for ambiguity.
\(^2\) Only the case $p = 2$ is proved in [4]; see [2, p. 1053] for all $p \geq 1$. The numbers in brackets refer to the bibliography at the end of the paper.
\(^3\) It is easy to give examples for which the individual ergodic theorem does hold while $L_p(m)$ is not transformed into itself. Such an example for instance is given if $T$ is periodic while $m$ is non-atomic and $T^{-1}$ is singular.

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the negative by constructing for each given \( p \geq 1 \) an example of a transformation of a measure space \((S, m)\) onto itself for which (1) the individual ergodic theorem holds, (2) the mean ergodic theorem does not hold in \( L_p(m) \), and (3) every function in \( L_p(m) \) is transformed into a function belonging to \( L_p(m) \).

**Remark.** Even though the individual ergodic theorem does not imply the mean ergodic theorem with respect to the original measure \( m \) it is known [2, p. 1059] that in case the individual ergodic holds, it is possible to introduce a new measure \( \mu \) defined on the measurable sets of \( S \) such that \( \mu \) has no more null sets than \( m \) and \( \mu \) is also invariant under \( T \). It follows then from the statements made above that the mean ergodic theorem holds in \( L_p(\mu) \) for every \( p \geq 1 \).

**The example for \( p = 1 \).** Let \( S \) be the totality of all points on the circumferences \( c_1, c_2, \ldots \) of a sequence of circles. Let the length of \( c_n \) be \( 1/2^n \). Let the measure \( m \) and the family of measurable sets in \( S \) be the obvious ones determined by the Lebesgue measure on each of the circumferences. On each circumference \( c_n \) we fix the polar coordinates \( \rho = (1/2^n \cdot 2\pi) \cdot e^{i\theta} \). Let \( x = \theta/2\pi \). Let us divide \( c_n \) into \( 2n + 2 \) arcs, the end points of the arcs being \( x = 0, x = 1/2, \) and \( x = \pm 1/2^{k+1}, k = 1, 2, \ldots, n \). The arcs are

\[
A_{nk} = \left[ \frac{1}{2^k}, \frac{1}{2^{k+1}} \right],
\]

\[
A_{n,n+1} = \left[ \frac{1}{2^{k+1}}, 1 \right],
\]

while if \( n + 2 \leq k \leq 2n + 2 \), \( A_{nk} \) is the reflection in \( \theta = 0 \) of \( A_{n,2n+3-k} \).

We define \( T \) as follows: For the points of \( A_{nk} \) let \( T \) be the unique transformation given by \( x' = ax + b, a > 0 \), which transforms \( A_{nk} \) onto \( A_{n,k+1} \) for \( k = 1, \ldots, 2n - 1 \), and \( A_{nk} \) onto \( A_{n,1} \) for \( k = 2n + 2 \). \( T \) is clearly a 1:1 point transformation of \( S \) onto itself. Moreover it is easily seen that \( T \) is measurable and pointwise periodic with the period of \( 2n + 2 \) for the points of \( c_n \). \( T \) also satisfies the following two conditions:

1. \( m(T^{-1}A) \leq 2m(A) \) for every measurable set \( A \) in \( S \).
2. The set of ratios

\[
R_A^n = \frac{1}{2n + 2} \sum_{i=0}^{2n-1} \frac{m(T^{-i}A)}{m(A)}
\]

where \( n = 1, 2, \ldots \) and \( A \) varies over all measurable sets \( A \subseteq S \) is not a bounded set of numbers. (1) follows from the fact that \( T^{-1} \) is de-
termined by a linear transformation with a stretching factor of at
most 2. (In fact, it is either 1/2, 1, or 2.)

To prove (2) consider the sequence of sets \( A_{n,n+1} \). Since
\[
\bigcup_{i=0}^{2^n-1} m(T^{-i}A_{n,n+1}) = c_n
\]
we have that
\[
\frac{1}{2n+2} \sum_{i=0}^{2^n-1} m(T^{-i}A_{n,n+1}) = \frac{1}{2n+2} \cdot \frac{1}{2^n}.
\]

On the other hand \( m(A_{n,n+1}) = (1/2^{n+1}) \cdot (1/2^n) \) and hence
\[
R^n_{A_{n,n+1}} = \frac{2^{n+1}}{2n+2},
\]
which is an unbounded sequence.

We can now show that the above example satisfies the required
conditions specified in the introduction.

(a) The individual ergodic theorem holds. In fact let \( f(y) \) be any
real-valued function defined on \( S \), then since \( T \) is pointwise periodic
the sequence \( \{ F_h(y) \} \) converges to a finite limit for every \( y \), that is,
the individual ergodic theorem holds for every real-valued function
defined on \( S \). A fortiori it holds (with respect to \( m \)).

(b) \( L_1(m) \) is transformed into itself by \( T \): In fact, it can be easily
seen that \( \bar{m}(A) = m(T^{-1}A) \) is a completely additive non-negative set
function (that is, a measure) defined on the measurable sets of \( S \). It
can also be shown by considering approximating sums to the integrals
that if \( f \in L_1(m) \) then
\[
\int_S |f(Ty)| \, dm = \int_S |f(y)| \, d\bar{m}.
\]

By (1), \( \bar{m}(A) \leq 2m(A) \) for every measurable set \( A \) and hence
\[
\int_S |f(y)| \, d\bar{m} \leq 2 \int_S |f(y)| \, dm < \infty
\]
from which follows that \( f(Ty) \in L_1(m) \), that is, \( L_1(m) \) is transformed
into itself by \( T \).

(c) The mean ergodic theorem does not hold in \( L_1(m) \). To prove
this statement we use the following result due to Miller and Dunford
[3, p. 539]: Suppose that the mean ergodic theorem did hold in
\( L_1(m) \); then there would exist a positive constant \( c \) independent of \( A \)
and \( h \) such that
\[
\frac{1}{h} \cdot \sum_{i=0}^{h-1} m(T^{-i}A) < c \cdot m(A)
\]
for all measurable sets $A$ and $h = 1, 2, \cdots$. But (i) is in contradiction with (2) above. Hence the mean ergodic theorem does not hold in $L_1(m)$.

It is possible to prove the last statement also directly by exhibiting functions in $L_1(m)$ for which the mean ergodic theorem does not hold in $L_1(m)$. In fact let $\tilde{f}(y)$ be defined as follows: $\tilde{f}(y) = 2^n$ on $A_{n,n+1}$, $n = 1, 2, \cdots$, $\tilde{f}(y) = 0$ everywhere else on $S$, then $\tilde{f}(y) \in L_1(m)$ but $\{F_h(y)\}$ is not convergent in $L_p(m)$, if it were then the limit function $\tilde{f}^*(y)$ of $\{F_h(y)\}$ would have to belong to $L_1(m)$. But $\tilde{f}^*(y)$ is seen to be equal to $(1/2n+2) \cdot 2^n$ for $y \in c_n$. We have

$$\int_S |\tilde{f}^*(y)| \, dm = \sum_{n=1}^{\infty} (1/2n + 2)$$

which is a divergent series, that is, $\tilde{f}^* \not\in L_1(m)$ and hence $\{F_h(y)\}$ is not convergent in $L_1(m)$.

**The example for $p \geq 1$.** Let $p$ be a fixed integer $\geq 1$. Let $S$ be the same sequence of circumferences $c_1$, $c_2$, $\cdots$ as before. We divide each $c_n$ into $2(n \cdot 2^p - n + 1)$ arcs, the end points being $x = 0$, $x = 1/2$ and $x = r/2^{k+1}$, $k = 1, 2, \cdots$, $n$, $r = 1, 2, \cdots$, $2^p - 1$. Again we define $T$ by the transformation given by $x' = ax + b$, $a > 0$, which transforms each arc into the next adjacent one.

$T$ is again seen to be a 1-1 measurable pointwise periodic transformation of $S$ onto itself with the period $2(n \cdot 2^p - n + 1)$ for the points of $c_n$. As before it follows that (a) the individual ergodic theorem holds. (b) $L_p(m)$ is transformed into itself since there is a bound (the bound being $2^p$) on the stretching factor of $T^{-1}$. (c) The mean ergodic theorem does not hold in $L_p(m)$. To prove this last statement we use the following generalization of Miller and Dunford’s result stated above: Let $t$ be any real number $\geq 1$, then if the mean ergodic theorem holds in $L_t(m)$ there exists a constant $C$ independent of $A$ and $h$ such that

$$(ii) \quad \left[ \frac{1}{h} \sum_{i=0}^{h-1} m(T^{-i}A) \right]^{1/t} < C \cdot m(A).$$

The proof is almost the same as for the special case $t = 1$. If, however, we consider the sequence of sets $A_{nq}$, where $q = n(2^p - 1) + 1$ and where the enumeration of the arcs on each $c_n$ is analogous to that used in the case $p = 1$, we can easily see that the sequence of ratios

$$R_n = \left[ \frac{1}{2q} \sum_{i=0}^{2q-1} m(T^{-i}A_{nq}) \right]^{p} / m(A_{nq})$$
is an unbounded sequence. This is in contradiction with (ii) for $t = p$. Hence the mean ergodic theorem does not hold in $L_p(m)$. Again the statement made in (c) may be proved directly by exhibiting functions in $L_p(m)$ for which the mean ergodic theorem does not hold in $L_p(m)$.

Let $p_1$ be any fixed number not less than 1. Let $p$ be the first integer not less than $p_1$. Then the example constructed above for $p$ is also a valid example for $p_1$, for the individual ergodic theorem clearly holds and $L_{p_1}(m)$ is transformed into itself for the same reasons as before, while it follows from the fact that $m(A) \leq 1$ for every measurable set $A$ and the fact that $p \geq p_1$ that the same sequence of sets which violates (ii) for the case $t = p$ also violates (ii) for $t = p_1$. Hence the mean ergodic theorem does not hold in $L_{p_1}(m)$.

BIBLIOGRAPHY