ON A PROBLEM OF MAX A. ZORN

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1. Introduction. Max A. Zorn has proved the following theorem.

**Theorem.** If every substitution $x = at$, $y = bt$ in which $a$ and $b$ are complex numbers transforms $\sum a_i x^i y^j$ into a power series with a non-vanishing radius of convergence, the series $\sum |a_i x^i y^j|$ converges for sufficiently small $|x|$ and $|y|$.

He has also suggested the following problem. If $X^a x^a y^b$ is a power series which is transformed by every substitution of convergent power series $\sum a_i x^i$ and $\sum b_i y^j$ with real coefficients for $x$ and $y$ into a convergent power series in $t$, is the double series $\sum a_i x^i y^j$ convergent?

The answer is yes. In fact, Zorn’s theorem itself holds even when the coefficients $a$ and $b$ are restricted to take only real values. We can obtain a proof quite directly by Zorn’s method, if we use an estimate for the coefficients of homogeneous polynomials in real variables.

2. Homogeneous polynomials in real variables. We shall prove a lemma which may easily be extended to the case of many variables.

**Lemma.** Let $P(x, y) = \sum_{i+j=n} a_{ij} x^i y^j$ be a homogeneous polynomial in real variables. If $|P(x, y)| \leq M$ for $|x - x_0| \leq 2\delta$, $|y - y_0| \leq 2\epsilon$, then $|a_{ij} x^i y^j| \leq M$.

**Proof.** Set $x = x_0 + \delta (\xi + \xi^{-1})$. Then $\xi^n P(x, y) = \sum a_{ij} \xi^i (\xi x)^j y^j$ is a polynomial in $\xi$ whose absolute value does not exceed $M$ when $\xi$ moves on the unit circle of the Gaussian plane. By Cauchy’s inequality of function theory, and considering the coefficients of $\xi^k$ in $\xi^n P(x, y)$, we have

$$\left| \sum_{j=0}^{k} a_{ij} x^{i} y^{j} \right| \leq M,$$

where $0 \leq k \leq n$, $i + j = n$, and $c_i$ is the coefficient of $\xi^{k-i}$ in $(\xi x)^i$.

Again set $y = y_0 + \epsilon (\eta + \eta^{-1})$ and apply the Cauchy inequality to the constant term of $\eta^k \sum_{j=0}^{k} a_{ij} c_i x^j y^j$. We have

$$|a_{lk} x^k| \leq M,$$

where $l + k = n$ and $c_l$ equals $\delta^l$ by definition. This completes our proof.

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3. **Proof of Zorn’s theorem in the real case.** Now we can follow Zorn’s method directly.

**Proof.** Let \( P_n(x, y) = \sum_{i+j=n} a_{ij} x^i y^j \). The set \( D \) of vectors \( (x, y) \) for which \( \sum P_n(x, y) \) converges is of the second category. For every vector is in \( mD \) for some positive integer \( m \). If \( D \) were of the first category, the set \( mD \) and therefore the two-dimensional Euclidean space would be the same character.

By virtue of the continuity of the functions \( P_n \) there will exist a square \( |x-x_0| \leq 2p, \quad |y-y_0| \leq 2p, \quad p > 0 \) and an \( M \) such that \( |P_n(x, y)| \leq M \) holds in the square for all \( n \). Then by our lemma \( |a_{ij} b^{i+j}| \leq M \). Hence for \( |x|, |y| \leq p/2 \), we have

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|a_{ij} x^i y^j| \leq M/2^{i+j}
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which establishes the absolute convergence of the double series.

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\( mD \) is the set of \( (mx, my) \) where \( (x, y) \subseteq D \).