INTEGRAL EXTENSIONS OF A RING
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Introduction. Let $R$ be a commutative ring with a unit element and let $a, b \subseteq R$.

Definitions. (1) $a$ and $b$ are said to be coprime in $R$ if $\tau \subseteq R$, $\tau/a$, $\tau/b$ implies $\tau/1$.

(2) A ring $R'$ is called an integral extension of $R$ if $R' \supseteq R$ and $a = br$, $a, b \subseteq R$, $r \subseteq R'$ implies there exists an element $\bar{r} \subseteq R$ such that $a = b\bar{r}$.

(3) $a$ and $b$ are said to be absolutely coprime if they are coprime in every extension $R'$ of $R$. In this paper it is shown that to every set of ideals of a commutative ring there exists an extension of the ring such that every ideal of the set is the intersection of the ring and a principal ideal of its extension. This is the main result and is given in Theorem 2. In a particular case of Theorem 2 it is shown in Theorem 1 that $a, b \subseteq R$ are absolutely coprime if and only if there exist elements $x, y \subseteq R$ such that $ax + by = 1$.

Similar results for algebraic integers are well known [1]. In the special case where the domains considered are completely integrally closed and the ideals have finite bases, a different extension fulfilling the conditions of Theorem 2 was obtained by Kronecker [2].

The extension of $R$. An extension of $R$, in the sense of this paper, may be obtained in the following manner. We first form the ring $R(u)$ by adjoining to $R$ the elements $u^n$, $n = \pm 1, \pm 2, \cdots$, transcendental over $R$ and such that $u^n a = au^n$, $a \subseteq R$. Let $a$ be the ideal generated by the set $A$ of elements $a, b, \cdots$ of $R$. Then the subring $R'$ of $R(u)$ consisting of all "polynomials"

$$a_m u^{-m} + a_{m+1} u^{-m+1} + \cdots + a_1 u^{-1} + a_0 + a_1 u + \cdots + a_n u^n$$

with $a_i \subseteq R$ and $a_r \subseteq a^r$, $r > 0$, is an integral extension of $R$ for $R' \supseteq R$. Moreover if $c = dh$, $c, d \subseteq R$, $h \subseteq R'$, then

$$h = e_m u^{-m} + \cdots + e_1 u^{-1} + e_0 + e_1 u + \cdots + e_n u^n$$

$e_i \subseteq R$, $e_r \subseteq a^r$. Multiplying by $d$ we have

$$c = de_m u^{-m} + \cdots + de_1 u^{-1} + de_0 + de_1 u + \cdots + de_n u^n.$$
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Since \( c \subset R \), all terms involving \( u \) must vanish. Hence \( c = de_0 \) and \( R' \) is an integral extension of \( R \).

Results.

Theorem 1. \( a, b \subset R \) are absolutely coprime if and only if there exist elements \( x, y \subset R \) such that \( ax + by = 1 \).

Clearly the condition is sufficient. Use the above extension \( R' \) of \( R \) where \( A \) consists of \( a \) and \( b \). In \( R' = R(au^{-1}, bu^{-1}, u) \), \( u \) is a common divisor of \( a \) and \( b \). If \( a \) and \( b \) are absolutely coprime, \( u \) must be a unit divisor. Hence

\[ u^{-1} = \alpha_{-m} u^{-m} + \cdots + \alpha_{-1} u^{-1} + \alpha_0 + \alpha_1 u + \cdots + \alpha_n u^n \]

\( \alpha_i \subset R, \alpha_{-i} \subset R' \). Since \( u \) is transcendental over \( R \) all terms in the sum must vanish except \( \alpha_{-1} u^{-1} \). But \( \alpha_{-1} \subset a \) and hence is of the form \( ax + by \). Multiplying by \( u \) we have

\[ 1 = ax + by. \]

Remark. If, \( x, y \) are elements of \( R \) such that

\[ ax + by = 1 \]

then the pair \( x, y \subset R \) is also a solution if and only if \( x = \hat{x} + b\mu, \ y = \hat{y} - a\nu \) where \( ab\mu = ab\nu \).

Substitution shows the condition to be sufficient. Moreover if \( ax + by = 1, \ a\hat{x} + b\hat{y} = 1 \) then \( a(x - \hat{x}) = b(\hat{y} - y) \) and so \( a\hat{x}(x - \hat{x}) = b(\hat{y} - y)x \). Adding \( b\hat{y}(x - \hat{x}) \) to each side of the last equation we have

\[ x - \hat{x} = b(\hat{y} - y)\hat{x} + b\hat{y}(x - \hat{x}) = b\mu. \]

Similarly \( y - \hat{y} = -a\nu \).

But \( a(x - \hat{x}) = b(\hat{y} - y) \). Hence

\[ ab\mu = ab\nu. \]

Lemma. If \( a \) is an ideal in \( R \), then there exists an extension \( R' \) of \( R \) such that \( a \) is the intersection of \( R \) and a principal ideal of \( R' \).

Consider the extension \( R' \) of \( R \) with \( a = A \). In \( R' \) the ideal \((u)\) is principal and contains \( a \) and no other elements of \( R \), for every element of \( a \) is obtained from the set of products \( au^{-1} x u \). Also for \( \lambda \subset R' \) suppose \( \lambda u = c, c \subset R \). Then \( \lambda = cu^{-1} \). Hence \( c \subset a \) and \( a = R \cap (u) \).

Theorem 2. To every set of ideals of \( R \) there exists an extension \( R' \) of \( R \) such that every ideal of the set is the intersection of \( R \) and a principal ideal of \( R' \).
To each ideal $a$ in the set let there correspond an element $u_a$ transcendental over $R$. Form the ring $R(u)$ by adjoining $u_a^* a$ to $R$ where $u_a^* a = au_a^*$, $a \subset R$, $n = \pm 1, \pm 2, \cdots$. The subring $R'$ of $R(u)$ consisting of the "polynomials"

$$\sum a_{r_1 r_2 \cdots r_n} u_1 u_2 \cdots \in R,$$

with the condition that $a_{r_1 r_2 \cdots r_i}$ belong to $a_i^{-r_i}$ if $r_i$ is negative, is an integral extension of $R$. This follows by the method demonstrated above. For if $l = mn$, $l, m \subset R$, $n \subset R'$ then

$$n = \sum \beta_{r_1 r_2 \cdots r_n} u_1 u_2 \cdots$$

where $\beta_{r_1 r_2 \cdots r_n}$ are certain $a_{r_1 r_2 \cdots r_n}$. Hence

$$l = mn = \sum m\beta_{r_1 r_2 \cdots r_n} u_1 u_2 \cdots.$$

But since the indeterminates $u_i$ are transcendental over $R$ and $l \subset R$, all terms in the sum, except the constant term $m\beta_0$, must vanish. Hence $b = m\beta_0$.

We proceed as in the lemma. Consider the principal ideal $(a)$. All of the elements $a^{(i)}$ of $a$ may be obtained as products $a^{(i)} u_a^{-1}. u_a$. Moreover only the elements $a^{(i)} \subset R$ may be so obtained for suppose $v \subset R'$ and $vu_i = d \subset R$. Then $v = du_i^{-1}$. Hence $a = R \setminus (u_i)$.

Remark. In the case of non-commutative rings a result analogous to Theorem 2 holds for two-sided ideals.

References


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