APPROXIMATION BY CURVES OF A UNISOLVENT FAMILY

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The aim of this paper is to extend theorems on the best approximation of a given function by a polynomial of a given degree, or more generally by a linear combination of given functions, first to the case where the approximating function is to be taken from a more general family of functions, which satisfies the requirements of continuity, solvence and unisolvence, to be specified below, and secondly to a class of related geometrical problems, for which that of approximating a curve, or six given points, by an ellipse may serve as an example. The best approximation is said to be furnished by that curve for which the maximum distance between corresponding points of the approximating and the approximated curves is as small as possible.[1]

Let (S) be a family of curves S represented by one-valued functions $y=S(x)$, $-1\leqq x\leqq 1$. The family (S) is assumed to be n-parametric, solvent, unisolvent, and continuous; more explicitly we assume:

1. *Solvence*: for any n values $x_1, \cdots, x_n$ with $-1\leqq x_1< \cdots <x_n\leqq 1$ and arbitrary real numbers $y_1, \cdots, y_n$ there exists a function S of (S) with $S(x_i)=y_i$, $i=1, \cdots, n$;

2. *Unisolvence*: only one such function exists, in the extended sense that not only, for any two different functions S_0 and S_1 of (S), S_0-S_1 has less than n roots (zeros), but also that this is true if any root x with $|x|<1$ for which S_0-S_1 does not change sign between $x-\epsilon$ and $x+\epsilon$ is counted as two roots;

3. *Continuity*: $S(x)=S(x; x_1, \cdots, x_n; y_1, \cdots, y_n)$ is a continuous function of $x, y_1, \cdots, y_n$.

It follows that there cannot exist $n+1$ values $-1\leqq x_0< \cdots <x_n \leqq 1$ for which S_0-S_1 has "alternating signs," that is, is alternatingly non-negative and non-positive.

For any curve S, $\sigma=\sigma(S)$ shall denote the supremum of the values

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[1] For the approximation by linear systems of functions see S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variables réelle*, Paris, 1926, Chap. 1. The approximation of systems of points in space and to some extent also of functions of several variables by linear functions and other polynomials is considered by P. Kirchberger, *Ueber Tchebychefsche Annaeherungsmethoden*, Math. Ann. vol. 57 (1903) pp. 509–540. It would be desirable to extend the method of the present note to the case of several independent variables.

According to Kirchberger loc. cit. p. 510, the approximation with least maximal deviation was first considered by Poncelet, and more systematically by Chebyshev.

$|S(x)|$; because of the continuity of S in the closed interval $-1 \leqq x \leqq 1$ this supremum is actually attained. We define furthermore the *oscillation number* $v = v(S) \geqq 0$ as one less than the maximum number of values $x_0 < \cdots < x_v$ such that $|S(x_0)| = \sigma$, $S(x_i) = -S(x_{i-1})$, $i = 1, \cdots, v$. If the number of such values is not bounded then $v(S) = \infty$.

A *nearest* curve S_0 of a family (S) to $y=0$ will be a curve for which $\sigma_0 = \sigma(S_0)$ is as small as possible. The existence, uniqueness and properties of S_0 are given in the following three theorems.

THEOREM I. *Under the above assumptions on* (S) *there exists a curve* $y = S_0(x)$, *and* $v_0 = v(S_0) \geqq n$.

THEOREM II. *For every other curve* S_1 *of* (S) *there are no values* $-1 \leqq x_0 < \cdots < x_n \leqq 1$ *with* $|S_1(x_i)| \geqq \sigma_0$, $i = 0, \cdots, n$, *and alternating signs of* $S_1(x_i)$.[2] *In particular* $v(S_1) < n$, *so that* S_0 *is unique and characterized by* $v(S_0) \geqq n$.

THEOREM III. *Also when* x *is restricted to* $n+1$ *values* $-1 \leqq x_0 < \cdots < x_n \leqq 1$ *with* $|S_0(x_i)| = \sigma_0$, $i = 0, \cdots, n$, *and alternating signs of* $S_0(x_i)$, S_0 *is the nearest curve to* $y=0$. *Moreover* S_0 *is the only curve of* (S) *with equal and alternating ordinates at the abscissae* x_i.

Theorems II and III follow immediately by remarking that if a curve $S_1 \neq S_0$ of (S) were situated in contradiction to either of them, $S_1 - S_0$ would have alternating signs for $x_0, \cdots, x_n$.

To prove Theorem I we observe first that since $\sigma(S)$ is a non-negative and continuous function $\sigma(y_1, \cdots, y_n)$ of the values of S for any n fixed numbers $-1 \leqq x_1 < \cdots < x_n \leqq 1$, and since the functions S with $\sigma \leqq N$ are bounded and form a closed family, the infimum $\sigma_0 \geqq 0$ of σ is in fact attained for a certain curve S_0.

Now if $v_0 < n$ (and thus $\sigma_0 > 0$), then let Ξ be the closed set of all roots of $|S_0(x)| = \sigma_0$, and choose $n-1$ values ξ_i, $i = 1, \cdots, n-1$, such that:

1. Each of the v_0 open intervals between consecutive numbers of Ξ that give alternating signs to S_0 contains one, or an odd number, of the values ξ_i;

2. Each of the other open intervals between consecutive numbers of Ξ (there may be 0 or a finite or infinite number of such intervals) contains none, or an even number, of the values ξ_i;

3. A half-open interval between -1 (inclusive) and the smallest number of Ξ, or 1 (inclusive) and the greatest number of Ξ, contains none, or any number, of the values ξ_i;

[2] This contains as a special case a result of de la Vallée-Poussin and its generalization by Bernstein loc. cit. p. 6.

4. Finally $\xi_{n-1}=1$, if

(*) $\qquad -1$ and 1 belong to Ξ and $n-1-v_0$ is odd.

Such a choice is obviously possible.

We now consider the curves S' of (S) for which $S'(\xi_i)=S_0(\xi_i)$, $i=1, \cdots, n-1$; because of the unisolvence assumed they meet only for $x=\xi_i$, and cross each other (that is, their difference changes sign) there. Due to the solvence of (S), S' may be chosen so that $S'(\xi)=S_0(\xi)(1-\epsilon)$ for a given value ξ of Ξ (different from 1 if (*) holds). For small $\epsilon>0$, S' will be near S_0 everywhere, because of the continuity of (S), and owing to the choice of the ξ_i we have $|S'|<\sigma_0$ for every value of Ξ (different from 1 if (*) holds) and therefore everywhere. Hence $\sigma(S')<\sigma_0$, against the definition of σ_0 (in case (*) holds we obtain $|S'(-1)|<\sigma_0$ and apply the same procedure to S' instead of S_0).[3] Thus the oscillation number v_0 must be at least n. Q.E.D.

The approximation of a given function $f(x)$, continuous for $-1\leqq x\leqq 1$, by a function of the family (S) has analogous properties; to see this replace every function S by $S-f$ (which does not affect the conditions imposed on the family) and apply Theorems I, II and III.

More generally the strip $|x|\leqq 1$, $|y|<\infty$ may undergo a topological mapping, the distance between the images of (x, y) and (x, y') being defined as $|y'-y|$.

If all functions S have the period 2, then the lines $x=-1$ and $x=1$ may be identified, and we obtain an approximation of a closed curve on a cylinder or within an annulus.

The extension of the theorems to approximations on the whole axis of real numbers, or in an open interval, requires additional assumptions.

The proof of Theorems I–III remains the same if (every $S(x)$ being still defined for $-1\leqq x\leqq 1$) $\sigma(S)$ denotes the supremum of $|S(x)|$ for values x belonging to a given closed partial set X of $-1\leqq x\leqq 1$ that contains at least $n+1$ numbers.

In the case of a finite set $X=(x_0, \cdots, x_m)$ the nearest curve can be found by determining, for every $n+1$ numbers of X, the one curve whose ordinates at these abscissae are equal but alternating in sign, and choosing the best one from among the finite number of curves obtained.

The related geometrical problem of finding, from a unisolvent

[3] In this case also, a single application of the procedure is sufficient: fix only $\xi_1, \cdots, \xi_{n-2}$ and put $S'(\xi)=S_0(\xi)(1-\epsilon)$ for $\xi=-1$ and $\xi=1$. There cannot be $n-1$ common points of S_0 and S' within $-1\leqq x\leqq 1$, since $n-1-v_0$ is odd.

n-parametric family (S) of plane curves, the nearest curve S_0 to a finite number of given points $P_0, \cdots, P_m$, $m \geqq n$, can be treated similarly. The family (S) is supposed to contain, within a certain domain of the real plane, through every n points one and only one curve, depending continuously on the situation of the n points. The nearest curve S_0 (that one for which the greatest distance $\sigma(S)$ from it of one of the points P_i is as small as possible), is again to be chosen from those curves for which $n+1$ from among the points P_i have equal distances from the curve. Indeed, if the number of approximated points P_k with maximal distance from S_0 were less than $n+1$, a small change of the curve could be effected (because of the solvence and continuity of the family) diminishing the maximal distance.

Owing to the unisolvence we can also prove by the same method of fixing $n-1$ points of the approximating curve as in the proof of Theorem I that $(n+1)/2(\pm 1/2)$ of the points P_k are on either side of S_0, with alternating nearest points Q_k on S_0, provided that there are exactly $n+1$ points P_k and that to every P_k there exists only one nearest point Q_k. In the general case it is impossible to divide S_0 by $n-1$ points into closed arcs, alternatingly called positive and negative, so that every point P_k on one side of S_0 has one of its nearest points Q_k on a positive arc, and every point P_k on the other side of S_0 has one of its nearest points Q_k on a negative arc. The value of $2\sigma(S_0)$ may be called the *breadth* of the given point set (P_i) with regard to the unisolvent family (S).[4]

As examples of unisolvent families we mention the family of all straight lines, the family of all conics, the family of all circles or more generally of all curves (positively) homothetic to a given, closed or infinite, convex curve that contains no straight segment. Each family has to be closed by including its limiting curves: points, straight lines and pairs of straight lines. Evidently a point cannot be a nearest curve, but a pair of straight lines can, and this possibility must be separately taken care of when determining the nearest conic to a point set (P_i). A straight line as S_0 behaves like a general member of the family.

It is easily seen that for a family of closed and bounded curves n will be odd.

The family of all parabolas is neither solvent nor unisolvent. Still since, in a bounded domain, parabolas that are sufficiently near to each other intersect only in $3=n-1$ points of the domain ("local

[4] For other properties of unisolvent families cf. Th. Motzkin, *Sur les arcs dont les courbes osculatrices ne se coupent pas*, C. R. Acad. Sci. Paris vol. 206 (1938) pp. 1700–1701.