A CONVEX METRIC FOR A LOCALLY CONNECTED CONTINUUM

R. H. BING

A topological space is metrizable if there is a distance function $D(x, y)$ such that if $x, y, z$ are points, then

(1) \( D(x, y) \geq 0 \), the equality holding only if $x = y$,
(2) \( D(x, y) = D(y, x) \) (symmetry),
(3) \( D(x, y) + D(y, z) \geq D(x, z) \) (triangle condition),
(4) \( D(x, y) \) preserves limit points.

By (4) we mean that $x$ is a limit point of the set $T$ if and only if for each positive number $\epsilon$ there is a point of $T$ at a positive distance from $x$ of less than $\epsilon$. We say that the metric $D(x, y)$ is convex if for each pair of points $x, y$ there is a point $u$ such that

(5) \( D(x, u) = D(u, y) = D(x, y)/2 \).

A subset $M$ of a topological space $S$ is said to have a convex metric (even though $S$ may have no metric) if the subspace $M$ of $S$ has a convex metric.

It is known [5] that a compact continuum is locally connected if it has a convex metric. The question has been raised [5] as to whether or not a compact locally connected continuum $M$ can be assigned a convex metric. Menger showed [5] that $M$ is convexifiable if it possesses a metric $D$ such that for each point $p$ of $M$ and each positive number $\epsilon$ there is an open subset $R$ of $M$ containing $p$ such that each point of $R$ can be joined in $M$ to $p$ by a rectifiable arc of length (under $D$) less than $\epsilon$. Kuratowski and Whyburn proved [4] that $M$ has a convex metric if each of its cyclic elements does. Beer considered [1] the case where $M$ is one-dimensional. Harrold found [3] $M$ to be convexifiable if it has the additional property of being a plane continuum with only a finite number of complementary domains.

We shall show that if $M_1$ and $M_2$ are two intersecting compact continua with convex metrics $D_1$ and $D_2$ respectively, then there is a convex metric $D_3$ on $M_1 + M_2$ that preserves $D_1$ on $M_1$ (Theorem 1). Using this result, we show that any compact $n$-dimensional locally connected continuum has a convex metric (Theorem 6). We do not

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1 Numbers in brackets refer to the references cited at the end of the paper.
answer the question: Does each compact locally connected continuum have a convex metric?²

In my paper Extending a metric it is shown [2, Theorem 5] that if $K$ is a closed subset of the metrizable space $S$ and $D_1$ is a metric on $K$, then there is a metric $D_2$ on $S$ that preserves $D_1$ on $K$. The following result is a modification of that result.

**Theorem 1.** If $M_1$ and $M_2$ are two intersecting compact continua with convex metrics $D_1$ and $D_2$ respectively, there is a convex metric $D_0$ on $M_1 + M_2$ that preserves $D_1$ on $M_1$.

**Proof.** Let $G(x)$ be the least upper bound of $D_1(p, q)$ for all points $p, q$ of $M_1 \cdot M_2$ such that $D_2(p, q) \leq x$. Then

$$G[D_2(p, q)] \geq D_1(p, q) \quad (p, q \text{ elements of } M_1 \cdot M_2).$$

Since $G(x)$ is a monotone nondecreasing function of $x$ that approaches zero as $x$ approaches zero, there is a function $F(x)$ ($x > 0$) such that $F(x)$ approaches zero as $x$ approaches zero, $F(x) \geq G(x)$, and the derivative of $F(x)$ with respect to $x$ is a continuous monotone non-increasing function greater than one.

If $C$ is an arc which lies except for possibly its end points in $M_2 - M_1 \cdot M_2$ and $\int C F'[D_2(p, M_1)]ds$ exists, where $F'(x)$ represents the derivative of $F(x)$ with respect to $x$, $p$ is a variable point of $C$, $s$ is the length along $C$ under $D_2$, and $D_2(p, M_1)$ is the greatest lower bound of $D_2(p, q)$ for all points $q$ of $M_1$, we define the length $L_0(C)$ of $C$ under $D_0$ to be

$$L_0(C) = \int C F'[D_2(p, M_1)]ds.$$

We note that $L_0(C)$ is not defined for all arcs $C$. However, if $r$ and $q$ are two points which lie on an arc $C$ with such a length $L_0(C)$, then we call $D_0(r, q)$ the greatest lower bound of $L_0(C)$ for all such arcs $C$ from $r$ to $q$. Since $F'(x) > 1$, $D_0(r, q) \geq D_2(p, q)$.

If $r$ and $q$ are two points of $M_1$ and $D_0(r, q)$ is defined, then

$$D_0(r, q) \geq D_1(r, q)$$

because for each positive number $\epsilon$ there is a curve $C$ from $r$ to $q$ whose interior lies in $M_1 - M_1 \cdot M_2$ such that

$$D_0(r, q) + \epsilon > \int C F'[D_2(p, M_1)]ds \geq \int C F'(s)ds$$

$$= F(\text{length } C \text{ under } D_2) \geq G[D_2(r, q)] \geq D_1(r, q).$$

The first “$\geq$” relationship in (8) follows from the facts that $F'(x)$ is a

² Since this paper was submitted, both E. E. Moise [8] and the author [9], working independently, have answered this question in the affirmative.
nonincreasing function and \( s \geq D_2(p, M_1) \) since the end points of \( C \) lie on \( M_1 \); the second "\( \geq \)" results from the facts that \( F(x) \geq G(x) \), length \( C \) under \( D_2 \geq D_2(r, q) \), and \( G(x) \) is a monotone nondecreasing function; the third "\( \geq \)" is a consequence of (6).

If \( q \) is a point of \( M_2 - M_1 \cdot M_2 \) and \( C \) is a shortest arc from \( q \) to \( M_1 \) under \( D_2 \), then

\[
L_0(C) = \int_{a}^{b} F'(D_2(p, M_1)) \, ds = \int_{a}^{b} F'(s) \, ds = F[D_2(q, M_1)].
\]

Hence, if \( p_1, p_2, \ldots \) is a sequence of points of \( M_2 - M_1 \cdot M_2 \) such that \( D_2(p_i, M_1) \) approaches zero as \( i \) increases without limit, then \( D_0(p_i, M_1) \) approaches zero as \( i \) increases without limit.

If \( p \) and \( q \) are two points of \( M_1 \), we define \( D_3(p, q) \) to be \( D_1(p, q) \); if \( p \) is a point of \( M_1 \) and \( q \) is a point of \( M_2 - M_1 \cdot M_2 \), we define \( D_3(p, q) \) to be the greatest lower bound of \( D_1(p, a) + D_0(a, q) \) for all points \( a \) of \( M_1 \); if both \( p \) and \( q \) are points of \( M_2 - M_1 \cdot M_2 \), we define \( D_3(p, q) \) to be the minimum of \( D_0(p, q) \) and the greatest lower bound of \( D_0(p, a) + D_1(a, b) + D_0(b, q) \), where \( a \) is a point of \( M_1 \) and so is \( b \). If \( p = q \), we define \( D_3(p, q) \) to be equal to zero.

The above definition of \( D_3(p, q) \) is equivalent to defining \( D_3(p, q) \) to be the greatest lower bound of the lengths of all arcs \( C \) from \( p \) to \( q \) where length in \( M_1 \) is measured under \( D_1 \) and length in \( M_2 - M_1 \cdot M_2 \) is measured under \( D_0 \). It follows from (7) that we need only consider those arcs \( C \) which intersect \( M_1 \) in a connected piece if at all.

Now the function \( D_3(p, q) \) may be shown to satisfy conditions (1), (2), (3), (4), and (5). Hence, it is a convex metric for \( M_1 + M_2 \) that preserves \( D_1 \) on \( M_1 \).

**Theorem 2.** If \( D_1 \) and \( D_2 \) are convex metrics for the intersecting compact continua \( M_1 \) and \( M_2 \) respectively and \( D_2 \geq D_1 \) on \( M_1 \cdot M_2 \), then for each positive number \( \epsilon \) there is a convex metric \( D_3 \) for \( M_1 + M_2 \) such that \( D_3 = D_1 \) on \( M_1 \), \( D_3 \leq D_2 \) on \( M_2 \), and the diameter under \( D_3 \) of each component of \( M_2 - M_1 \cdot M_2 \) is less than \( \epsilon \) plus twice the diameter under \( D_1 \) of the boundary with respect to \( M_1 + M_2 \) of this component.

**Proof.** Define \( E(p, q) \) to be the greatest lower bound of all sums of the type \( f(p, p_1) + f(p_1, p_2) + \cdots + f(p_n, q) \) where adjacent points of \( p, p_1, \ldots, p_n, q \) belong to the same one of the continua \( M_1, M_2 \) and \( f(p_i, p_j) \) is \( D_1(p_i, p_j) \) or \( D_2(p_i, p_j) \) according as \( p_i + p_j \) is or is not a subset of \( M_1 \). Since \( D_2 \geq D_1 \) on \( M_1 \cdot M_2 \), \( n \) need be no larger than 2 for \( E(p, q) \) to attain this greatest lower bound. The convex metric \( E \) on \( M_1 + M_2 \) preserves \( D_1 \) on \( M_1 \) and \( E \leq D_2 \) on \( M_2 \).

Let \( X \) be the set of all points \( p \) of \( M_2 - M_1 \cdot M_2 \) such that the
distance from $p$ to $M_1$ under $E$ is greater than one-half the diameter under $E$ of the boundary with respect to $M_1 + M_2$ of the component of $M_2 - M_1$. Let $n$ be a positive number so small that $\varepsilon/n$ is greater than twice the diameter of $M_2$ under $E$. If $C$ is a rectifiable (under $E$) arc in $M_1 + M_2$ from $p$ to $q$, define $L(C)$ to be the greatest lower bound of all sums of the type $f(p, p_1) + f(p_2, p_3) + \cdots + f(p_k, q)$ where $p_1, p_2, \cdots, p_k$ are points of $C$ and $f(p_i, p_j)$ is either $n$ times or 1 times the length under $E$ of the subarc of $C$ from $p_i$ to $p_j$ according as this subarc is or is not a subset of $X$. If $D_2(p, q)$ is defined to be the greatest lower bound of all such values $L(C)$, $D_2$ satisfies the conditions of Theorem 2.

**Theorem 3.** Suppose $M_2$ is a compact continuum with a convex metric, $M_2$ lies in a complete locally connected space $S$ with a metric $D$, each component of $S - M_2$ is of diameter under $D$ of less than $\theta$ and $M_1$ is a subcontinuum of $M_2$ with a convex metric $D_1$ such that

$$D_1(p, q) \leq D(p, q) \quad \text{if } D_1(p, q) > \theta \quad (p, q \text{ elements of } M_1).$$

For each positive number $\varepsilon$ there is a continuum $M_3$ containing $M_2$ and a convex metric $D_3$ for $M_3$ such that $D_3$ preserves $D_1$ on $M_1$ and the boundary of each component of $S - M_3$ is of diameter less than $\theta + \varepsilon$ under $D_3$.

**Proof.** By Theorem 1, there is a convex metric $D_2$ for $M_2$ that preserves $D_1$ on $M_1$. Let $n$ be an integer so large that

$$nD_2(p, q) > D(p, q) \quad \text{if } D(p, q) > \varepsilon/8 \quad (p, q \text{ elements of } M_2).$$

Let $X$ denote the collection of all pairs of points $(x, y)$ such that both $x$ and $y$ are points of the boundary of the same component of $S - M_2$ and

$$nD_2(x, y) > \theta + \varepsilon.$$

There is a finite subcollection $X'$ of $X$ such that for each element $(x, y)$ of $X$ there is an element $(x', y')$ of $X'$ such that both $x'$ and $y'$ are accessible from the same component of $S - M_2$ and

$$nD_2(x, x') + nD_2(y', y) < \varepsilon/2.$$

Let $C_1, C_2, \cdots, C_j$ be a finite collection of components of $S - M_2$ irreducible with respect to the property that for each element of $X'$ there is an integer $i$ less than or equal to $j$ such that both points of this element of $X'$ are accessible from $C_i$. There is a dendron $T_i$ ($i = 1, 2, \cdots, j$) such that $T_i$ lies except for
its ends in $C_i$ and the sum of the ends of $T_i$ is a finite subset $Y_i$ of the boundary of $C_i$ such that $Y_i$ contains all points of the sum of the elements of $X'$ that are accessible from $C_i$ and for each point $q$ of the boundary of $C_i$ there is a point $r$ of $Y_i$ such that $nD_2(r, q) < \epsilon/2$. Let $D(T_i)$ be a convex metric for $T_i$ such that if $p$ and $q$ are two end points of $T_i$, then

$$\theta + \epsilon/4 < D(T_i; p, q) < \theta + \epsilon/2. \tag{13}$$

Let $E$ be a metric for $M_2 + \sum T_i = M_3$ such that the distance between two points of $M_3$ under $E$ is the greatest lower bound of the lengths of arcs containing them where length is measured by $nD_2$ in $M_2$ and by $D(T_i)$ in $T_i$. If $(x, y)$ is an element of $X$, it follows from (12) and (13) that

$$E(x, y) < \theta + \epsilon. \tag{14}$$

The diameter of the boundary of each component of $S - M_3$ is less than $\theta + \epsilon$ under $E$, for suppose $p$ and $q$ are two points of this boundary; if $p + q$ is a subset of $M_2$, $E(p, q) < \theta + \epsilon$ by (11) and (14); if neither $p$ nor $q$ is a point of $M_2$, both belong to some $T_i$ whose diameter is less than $\theta + \epsilon/2$; if $p$ is an interior point of a $T_i$ which does not contain $q$, there is an end point $r$ of $T_i$ such that $nD_2(r, q) < \epsilon/2$ and then $E(p, q) \leq D(T_i; p, r) + nD_2(r, q) < \theta + \epsilon$.

We shall show that if $r$ and $s$ are two points of $M_1$, then $D_1(r, s) \leq E(r, s)$. Suppose this is not the case and that $rs$ is an arc from $r$ to $s$ in $M_3$ whose length is less than $D_1(r, s)$ under $E$. If $D_1(r, s) \leq \theta$, then $rs$ is a subset of $M_2$ alone and

$$E(r, s) = nD_2(r, s) \geq D_2(r, s) = D_1(r, s). \tag{15}$$

Suppose $rs$ is not a subset of $M_2$ and $p_1p_2, p_2p_4, \ldots, p_{2i-1}p_{2j}$ are the subarcs of $rs$ which lie except for their end points in $M_3 - M_2$ where $p_{2i-1}p_{2j}$ precedes $p_{2i+1}p_{2i+2}$ on $rs$ in the order from $r$ to $s$. Let $Z(t)$ be 0 or $t$ according as $t$ is less than $\epsilon/8$ or not. If $D_1(r, s) > \theta$, it follows from (9), (3), and (10) that

\[
D_1(r, s) \leq D(r, s) \leq D(r, p_1) + D(p_1, p_2) + \cdots + D(p_{2j}, s) \\
< D(r, p_1) + \theta + D(p_2, p_3) + \cdots + \theta + D(p_{2j}, s) \\
< Z[D(r, p_1)] + Z[D(p_2, p_3)] + \cdots + Z[D(p_{2j}, s)] \\
+ j\theta + (j+1)\epsilon/8 \\
\leq nD_2(r, p_1) + nD_2(p_2, p_3) + \cdots + nD_2(p_{2j}, s) \\
+ j(\theta + \epsilon/4) \\
\leq \text{length } rs \text{ under } E. \tag{16}
\]
It follows from (15) and (16) that $E(r, s) \geq D_1(r, s)$.

It follows from Theorem 2 that there is a convex metric $D_3$ for $M_3$ such that $D_3 = D_1$ on $M_1$ and $D_3 \leq E$. The boundary of each component of $S - M_3$ is of diameter less than $\theta + \varepsilon$ under $D_3$ because it is under $E$.

**Theorem 4.** Suppose $M$ is a compact locally connected continuum such that if $p$ is a point contained in an open subset $R_1$ of $M$, there is an open subset $R_2$ of $R_1$ containing $p$ such that the boundary of $R_2$ with respect to $M$ is a subset of a subcontinuum of $M$ with a convex metric. Then $M$ has a convex metric.

**Proof.** Let $F$ be a metric for $M$. We shall show that there is a collection of continua $M_1, M_2, \ldots$ in $M$ and a collection of metrics $D_1, D_2, \ldots$ such that:

(a) $M_{i+1}$ contains $M_i$.
(b) $D_i$ is a convex metric for $M_i$.
(c) $D_{i+1}$ preserves $D_i$ on $M_i$.
(d) Under $F$, each component of $M - M_i$ is of diameter less than $1/4^i$.
(e) Under $D_i$, the boundary (with respect to $M$) of each component of $M - M_i$ is of diameter less than $1/4^i$.
(f) Under $D_{i+1}$, the common part of the $M_{i+1}$ and each component of $M - M_i$ is of diameter less than $3/4^i$.

First, we show that if $\varepsilon$ is a positive number, there is a subcontinuum $W$ of $M$ with a convex metric such that each component of $M - W$ is of diameter less than $\varepsilon$ under $F$. By the Heine-Borel Theorem, we find that there is a finite collection $G$ of subcontinua of $M$ such that each element of $G$ has a convex metric, for each point $p$ of $M$ there is an element $g$ of $G$ such that $p$ belongs either to $g$ or to a component of $M - g$ of diameter under $F$ of less than $\varepsilon$, and the sum $W$ of the elements of $G$ is a continuum. Each component of $M - W$ is of diameter less than $\varepsilon$ under $F$ and it follows from Theorem 1 that $W$ has a convex metric.

Let $M_1$ be a subcontinuum of $M$ with a convex metric $E$ such that each component of $M - M_1$ is of diameter less than $1/4$ under $F$. A suitable multiple of $E$ gives a metric $D_1$ for $M_1$ which will satisfy conditions (b) and (e).

Let $n$ be an integer so large that

$$nF(p, q) > D_1(p, q) \quad \text{if} \quad D_1(p, q) > 1/17 \quad (p, q \text{ elements of } M_1).$$

There is a continuum $W$ in $M$ containing $M_1$ such that $W$ has a convex metric and the diameter of each component of $M - W$ is less than
By Theorem 3, there is a continuum $M_2$ with a convex metric $E$ such that $M_2$ contains $W$, $E$ preserves $D_1$ on $M_1$, and the diameter under $E$ of the boundary with respect to $M$ of each component of $M - M_1$ is less than $1/4^2$. Applying Theorem 2 we find that $M_2$ has a convex metric $D_2$ such that $M_2$ and $D_2$ satisfy conditions (a), (b), (c), (d), (e), and (f).

Similarly, there exist continua $M_3$, $M_4$, \ldots and convex metrics $D_3$, $D_4$, \ldots satisfying conditions (a), (b), (c), (d), (e), and (f). Let $D$ be a function of the pairs of points of $M$ such that $D(p, q)$ is the lower limit of $D_1(p_1, q_1), D_2(p_2, q_2), \ldots$ where $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ are sequences of points converging to $p$ and $q$ respectively and $p_i + q_i$ is a subset of $M_i$. We shall show that $D$ preserves limit points in $M$.

Let $\epsilon$ be a positive number and $n$ an integer so large that $3/4^n + 3/4^{n+1} + \ldots < \epsilon/4$. If $p$ and $q$ are points of the same component of $M - M_n$, then $D(p, q) < \epsilon/2$ by (f). If $p$ is a point of $M_n$, let $R$ be the set of all points $r$ of $M_n$ such that $D_n(p, r) < \epsilon/2$. The sum $V$ of $R$ and all components of $M - M_n$ that have a point of $R$ on their boundaries is an open subset of $M$ containing $p$ and if $q$ is a point of $V$, then $D(p, q) < \epsilon$. Hence, the set of points $q$ such that $D(p, q) > \epsilon$ is not a limit point of $p$.

If $V$ is an open subset of $M$ containing $p$, we shall show that there is a positive number $\epsilon$ such that $D(p, M - V) < \epsilon$. Let $R$ be an open subset of $V$ containing $p$ and $n$ an integer such that $F(R, M - V) > 3/4^n$. There is a positive number $\epsilon$ such that $D_n(r, s) > \epsilon$ if $F(r, s) > 1/4^n$. Since each component of $M - M_n$ is of diameter less than $1/4^n$ under $F$, each arc in $M$ from $R$ to $M - V$ contains points $r$ and $s$ of $M_n$ such that $F(r, s) > 1/4^n$. Hence, if $k$ is an integer bigger than $n$, $D_k(R, M_n, [M - V], M_k) > \epsilon$. Hence, $D(p, M - V) < \epsilon$.

We have shown that $D$ satisfies conditions (1), (2), and (4). As each $D_i$ satisfies conditions (3) and (5) and $D$ is the limit of $D_1, D_2, \ldots$, then $D$ satisfies these conditions. Hence, it is a convex metric for $M$.

**Theorem 5.** If $M$ is an $n$-dimensional locally connected compact continuum and $\epsilon$ is a positive number, there is a locally connected continuum $W$ in $M$ such that each component of $M - W$ is of diameter less than $\epsilon$, $W$ is $(n-1)$-dimensional if $n > 1$, and $W$ is a dendron (acyclic continuous curve) if $n = 1$.

**Proof.** An application of the Heine-Borel Theorem gives that there is an $(n-1)$-dimensional closed subset $H$ of $M$ such that each component of $M - H$ is of diameter less than $\epsilon$. If $n-1 > 0$, $H$ is
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contained in an \((n-1)\)-dimensional locally connected subcontinuum of \(M\) [6, Theorem 1]. If \(n-1=0\), a dendron in \(M\) contains \(H\) [7, Theorem 1].

**Theorem 6.** Each \(n\)-dimensional compact locally connected continuum has a convex metric.

**Proof.** If \(n=1\), Theorem 6 follows from Theorems 4 and 5 and the fact that a dendron has a convex metric. If \(n>1\), Theorem 6 follows from Theorems 4 and 5 and induction on \(n\).

**Definition.** A set \(S\) is said to be finite-dimensional if for each point \(p\) of \(S\) and each open subset \(R\) of \(S\) containing \(p\) there is an integer \(n\) and an open subset \(R'\) of \(R\) containing \(p\) such that the boundary of \(R'\) with respect to \(S\) is \(n\)-dimensional.

The following result may be established by using Theorems 4, 5, and 6.

**Theorem 7.** Each finite-dimensional compact locally connected continuum has a convex metric.

**References**


**University of Wisconsin**