REAL ROOTS OF DIRICHLET L-SERIES

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Let $k$ be a positive integer. Let $\chi$ be a real, non-principal character (mod $k$) and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the corresponding $L$-series, which converges uniformly for $R(s) \geq \epsilon > 0$. If it could be shown that uniformly in $k$ there is no real zero of $L(s, \chi)$ for

$$s \geq 1 - \frac{A}{\log k},$$

where $A$ is a constant, then the existing theorems on the distribution of primes in arithmetic progressions could be greatly improved (see [1]). Moreover by Hecke's Theorem (see [2]) it would follow that uniformly in $k$

$$L(1, \chi) > \frac{B}{\log k}$$

where $B$ is a constant. This would be a considerable improvement over Siegel’s Theorem (see [3]), and would lead to an improved lower bound for the class number of an imaginary quadratic field.

In the present paper, we shall show that for $2 \leq k \leq 67$, $L(s, \chi)$ has no positive real zeros. By combining this information with the results of [1], we infer very sharp estimates on the distribution of primes in arithmetic progressions of difference $k$ for $k \leq 67$.

The methods used for $k \leq 67$ certainly will suffice for many other $k$’s greater than 67. They may possibly suffice for all $k$, but we can find no proof of this.²

In [5], S. Chowla has considered the positive real zeros of $L(s, \chi)$, and shown that for many explicit $k$’s, no positive real zeros exist. However Chowla could not settle whether his methods would suffice

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¹ Numbers in brackets refer to the bibliography at the end of the paper.
² These methods have been tried on all $k \leq 227$ and it has been ascertained that except for the cases $k = 148$ and $k = 163$, $L(s, \chi)$ has no positive real zeros for $2 \leq k \leq 227$. Cases $k = 148$ and $k = 163$ are now being studied and any results obtained about them will appear in the Journal of Research of the National Bureau of Standards.
to handle the difficult cases $k = 43$ and $k = 67$. In [6], Heilbronn has shown that there exist values of $k$ for which Chowla's methods are certainly inadequate.

**THEOREM 1.** If $\chi$ is non-principal $(\text{mod } k)$ and $\chi(-1) = 1$, then for all $s$

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{2s(s+1) \cdots (s+2\alpha-1)}{4^\alpha(2\alpha)! \cdot k^{\alpha+2\alpha}} (2^s+2\alpha-1)\xi(s+2\alpha)$$

$$\cdot \sum_{n=1}^{[k/2]} \chi(n)(k-2n)^{2\alpha}.$$

**PROOF.** For $s > 1$, we have

$$L(s, \chi) = 2^s \sum_{N=0}^{\infty} \sum_{n=1}^{k-1} \frac{\chi(n)}{(2kN + 2n)^s}$$

$$= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \sum_{n=1}^{k-1} \chi(n) \left(1 - \frac{k-2n}{k(2N+1)}\right)^{-s}$$

$$= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \sum_{n=1}^{k-1} \chi(n) \left(1 + s \frac{k-2n}{k(2N+1)} \right)^{-s}$$

$$+ \frac{s(s+1)}{2!} \left(\frac{k-2n}{k(2N+1)}\right)^{2}$$

$$+ \frac{s(s+1)(s+2)}{3!} \left(\frac{k-2n}{k(2N+1)}\right)^{3} + \cdots \}$$

$$= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \left\{ \frac{s}{k(2N+1)} \sum_{n=1}^{k-1} \chi(n)(k-2n)$$

$$+ \frac{s(s+1)}{2!k^s(2N+1)^2} \sum_{n=1}^{k-1} \chi(n)(k-2n)^2 + \cdots \}$$

$$= \frac{s}{2k^{s+1}} \left( \sum_{N=0}^{\infty} \left(\frac{2}{2N+1}\right)^{s+1} \right) \sum_{n=1}^{k-1} \chi(n)(k-2n)$$

$$+ \frac{s(s+1)}{4(2!)k^{s+2}} \left( \sum_{N=0}^{\infty} \left(\frac{2}{2N+1}\right)^{s+2} \right) \sum_{n=1}^{k-1} \chi(n)(k-2n)^2 + \cdots$$

$$= \frac{s}{2k^{s+1}} (2^{s+1} - 1)\xi(s+1) \sum_{n=1}^{k-1} \chi(n)(k-2n)$$

$$+ \frac{s(s+1)}{4(2!)k^{s+2}} (2^{s+2} - 1)\xi(s+2) \sum_{n=1}^{k-1} \chi(n)(k-2n)^2 + \cdots.$$
Since \( \chi \) is non-principal, we have \( k>2 \), and so if \( k \) is even, we have \( \chi(\lceil k/2 \rceil) = \chi(k/2) = 0 \). Now since \( \chi(-1) = 1 \),
\[
\sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha}
\]
\[
= \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha} + \sum_{n=\lfloor k/2 \rfloor+1}^{k-1} \chi(n)(2n - k)^{2\alpha}
\]
\[
= \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha} + \sum_{n=k-\lfloor k/2 \rfloor}^{\lfloor k/2 \rfloor} \chi(n)(k - 2(k - n))^{2\alpha}
\]
\[
= \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha} + \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - n)(k - 2n)^{2\alpha}
\]
\[
= 2 \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha}.
\]
Similarly, we prove \( \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha+1} = 0 \).
Thus we infer that the equation stated is valid for \( s>1 \).
Now since
\[
\left| \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha} \right| \leq \frac{k}{2} (k - 2)^{2\alpha},
\]
we see that the series on the right converges absolutely and uniformly for all \( s \), and so our theorem follows by analytic continuation.

**Theorem 2.** If \( \chi \) is non-principal (mod \( k \)) and \( \chi(-1) = -1 \), then for all \( s \)
\[
L(s, \chi) = \sum_{\alpha=0}^{\infty} \frac{s(s + 1) \cdots (s + 2\alpha)}{4^\alpha(2\alpha + 1)!k^{s+2\alpha+1}} (2^{s+2\alpha+1} - 1)\zeta(s + 2\alpha + 1)
\]
\[
\cdot \sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^{2\alpha+1}.
\]
The proof is similar to the proof of Theorem 1.
Although these theorems hold for any non-principal \( \chi \), we shall use them only for real non-principal \( \chi \). We assume henceforth that \( \chi \) is real and non-principal. We let \( \Sigma_M \) denote
\[
\sum_{n=1}^{\lfloor k/2 \rfloor} \chi(n)(k - 2n)^M.
\]
For sufficiently large \( M \) (certainly for \( M \geq k \)), the initial term
\[
\chi(1)(k - 2)^M
\]
of $\Sigma_M$ dominates the remaining terms, and we infer that $\Sigma_M > 0$. If by good chance $\Sigma_M \geq 0$ for all $M \geq 1$, then by Theorem 1 or Theorem 2 we infer that $L(s, \chi) > 0$ for $s > 0$, and hence that $L(s, \chi)$ has no positive real zeros. For $k \leq 67$, this happens in a majority of cases.

When considering positive real zeros of $L(s, \chi)$ it suffices to restrict attention to primitive $\chi$'s (and to the $k$'s for which there are primitive $\chi$'s. See [4, §125]). For primitive $\chi$'s, $\Sigma_M \geq 0$ for $M \geq 1$ for each $k \leq 67$ except 43 and 67. Moreover for each such $k$, the proof of $\Sigma_M \geq 0$ is easily accomplished by grouping the terms in groups, each of which is non-negative. Typical such groups are:

I. $A^M - B^M$, where $A > B$.
II. $A^M - B^M - C^M$, where $A \geq B + C$.
III. $A^M - B^M - C^M + D^M$, where $A + D \geq B + C$.

For $k = 53$, there occurs the group $51^M - 49^M - 47^M + 45^M - 43^M + 41^M + 39^M - 37^M$, which we show to be non-negative by writing it as $(44+7)^M - (44+5)^M + (44+3)^M - (44+1)^M - (44-1)^M + (44-3)^M + (44-5)^M - (44-7)^M$, and expanding each term by the binomial theorem.

For $k = 43$ or 67, we have $\Sigma_3 < 0$, so that the series in Theorem 2 does not consist entirely of non-negative terms. However, we can show that the initial positive term outweighs the negative terms. We give the proof for $k = 67$, since the proof for $k = 43$ is similar and easier.

By the functional equation for $L(s, \chi)$ (see [4, §128]) it follows that if $L(s, \chi)$ has a zero $\rho$ with $1/2 < \rho < 1$, then it has a zero $\rho$ with $0 < \rho < 1/2$. As it is known that $L(s, \chi) > 0$ for $1 \leq s$, it suffices to prove $L(s, \chi) > 0$ for $0 \leq s \leq 1/2$. So we take $k = 67$ and $0 \leq s \leq 1/2$. By Theorem 2,

$$L(s, \chi) = \frac{2^{s+1} - 1}{67^s} \left\{ \frac{s^2(s + 1)}{M} \Sigma_1 + \frac{s(s + 1)(s + 2)}{3!(67)^3} \frac{2^{s+3} - 1}{4(2^{s+1} - 1)} \zeta(s + 3) \Sigma_3 + \cdots \right\},$$

where now $\Sigma_M = \sum_{n=1}^{33} \chi(n)(67 - 2n)^M$. For $s > 0$,

$$\zeta(s + 1) - \frac{1}{s} = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} - \int_1^{\infty} \frac{dx}{x^{s+1}} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^{s+1}} - \int_n^{n+1} \frac{dx}{x^{s+1}} \right\} > 0.$$
For $0 \leq s$

\[ \frac{s^2(s+1)}{67} \geq 1. \]

\[ \frac{s^{2+2s+1} - 1}{4^s(2s+1) - 1} < \frac{2s}{2 - 2^{-s}} \quad \text{and} \quad \frac{d}{ds} \left( \frac{2s}{2 - 2^{-s}} \right) > 0. \]

So for $0 \leq s \leq 1/2$

\[ \frac{s^{2+2s+1} - 1}{4^s(2s+1) - 1} \leq \frac{2(1/2)}{2 - 2^{-1/2}} < 0.77346. \]

Also

\[ \frac{(s + 1)(s + 2)}{3!} \leq \frac{(3/2) \cdot (5/2)}{3!} = \frac{5}{8}. \]

Since $\Sigma_3 = -102,845$, we infer

\[ \frac{s(s + 1)(s + 2)}{3!(67)^3} \frac{2^{s+3} - 1}{4(2s+1) - 1} \xi(s + 3) \Sigma_3
\]

\[ \geq - \frac{5}{8} \frac{1}{(67)^3} (0.77346) \xi(3)(102,845)
\]

\[ \geq - \frac{5}{8} \frac{(0.77346)(1.20206)}{300,763} 102,845
\]

\[ > - 0.199. \]

Now for $M \geq 1$,

\[ \Sigma_M = \{ (57 + 8)^M - (57 + 6)^M - (57 + 4)^M + (57 + 2)^M - 57^M + (57 - 2)^M - (57 - 4)^M - (57 - 6)^M + (57 - 8)^M \}
\]

\[ + \{ (43 + 4)^M - (43 + 2)^M - 43^M - (43 - 2)^M + (43 - 4)^M \}
\]

\[ + 37^M + 35^M + \ldots \]

\[ > - 57^M + \frac{M(M - 1)}{2!} 57^{M-2} \{ 2 \cdot 8^2 - 2 \cdot 6^2 - 2 \cdot 4^2 + 2 \cdot 2^2 \}
\]

\[ + \frac{M(M - 1)(M - 2)(M - 3)}{4!} 57^{M-4} \{ 2 \cdot 8^4 - 2 \cdot 6^4
\]

\[ - 2 \cdot 4^4 + 2 \cdot 2^4 \} + \ldots
\]

\[ - 43^M + \frac{M(M - 1)}{2!} 43^{M-2} \{ 2 \cdot 4^2 - 2 \cdot 2^2 \} + \ldots \]
\[ \geq - 57^M \left( 1 - \frac{16M(M - 1)}{57^2} \right) - 43^M \left( 1 - \frac{12M(M - 1)}{43^2} \right). \]

In particular, if \( \alpha \geq 2 \), then

\[ \Sigma_{2\alpha+1} \geq - 57^{2\alpha+1} \left( 1 - \frac{16(2\alpha + 1)2\alpha}{57^2} \right) - 43^{2\alpha+1} \left( 1 - \frac{12(2\alpha + 1)2\alpha}{43^2} \right) \]

\[ \geq - 57^{2\alpha+1} \left( 1 - \frac{320}{3249} \right) - 43^{2\alpha+1} \left( 1 - \frac{240}{1849} \right) \]

\[ \geq - 57^{2\alpha+1} \frac{2929}{3249} - 43^{2\alpha+1} \frac{1609}{1849}. \]

So for \( 0 \leq s \leq 1/2 \),

\[ \sum_{\alpha=2}^{\infty} \frac{s(s + 1) \cdots (s + 2\alpha)}{(2\alpha + 1)! (67)^{2\alpha+1}} \frac{2^{s+2\alpha+1} - 1}{4^{s+1} - 1} \xi(s + 2\alpha + 1) \Sigma_{2\alpha+1} \]

\[ \leq - \sum_{\alpha=2}^{\infty} \frac{s(s + 1) \cdots (s + 2\alpha)}{(2\alpha + 1)! (67)^{2\alpha+1}} \frac{2^{s+2\alpha+1} - 1}{4^{s+1} - 1} \xi(s + 2\alpha + 1) \]

\[ \cdot \left\{ \frac{57^{2\alpha+1} 2929}{3249} + \frac{43^{2\alpha+1} 1609}{1849} \right\} \]

\[ - \sum_{\alpha=2}^{\infty} \frac{s(s + 1) \cdots (s + 4)}{6! (67)^{2\alpha+1}} \frac{2^{s+2\alpha+1} - 1}{4^{s+1} - 1} \xi(5) \]

\[ \cdot \left\{ \frac{57^{2\alpha+1} 2929}{3249} + \frac{43^{2\alpha+1} 1609}{1849} \right\} \]

\[ \leq - \frac{63}{128} (0.77346)(1.03693) \sum_{\alpha=2}^{\infty} \left\{ \frac{57^{2\alpha+1} 2929}{67} \cdot \frac{1}{3249} \right\} \]

\[ + \left\{ \frac{43^{2\alpha+1} 1609}{67} \cdot \frac{1}{1849} \right\} \]

\[ \leq - \frac{63}{128} (0.77346)(1.03693) \left\{ \frac{57^{8} 4489}{67} \cdot \frac{2929}{1240} \cdot \frac{1}{3249} \right\} \]

\[ + \left\{ \frac{43^{8} 4489}{67} \cdot \frac{1609}{2640} \cdot \frac{1}{1849} \right\} \]

\[ > - 0.638. \]
By (1), (2), and (3), for $0 \leq s \leq 1/2$,
\[
L(s, \chi) \geq \frac{2^{s+1} - 1}{6^s} \{1.000 - 0.199 - 0.638\} \geq 0.163 \geq 0.0199.
\]
So $L(s, \chi) > 0$ for $0 \leq s$.

When $\chi(-1) = -1$, Theorem 2 opens up further interesting possibilities. When $s \to 0$, the first term of the series is bounded away from zero, while the remaining terms approach zero. Thus one can always infer $L(s, \chi) > 0$ for $0 \leq s \leq \varepsilon$, where $\varepsilon$ depends on $k$. Even for $\varepsilon$ as small as $\frac{A}{\log k}$, this would be a very worthwhile result, as remarked at the beginning of the paper.

For another possibility, let $s = 0$ and $-2$ in Theorem 2, and evaluate $L(0, \chi)$ and $L(-2, \chi)$ by the functional equation. We infer the known result
\[
L(1, \chi) = \frac{\pi}{k^{3/2}} \Sigma_1
\]
and the result
\[
L(3, \chi) = \frac{\pi^3}{6 k^{7/2}} \{k^2 \Sigma_1 - \Sigma_2\}.
\]
From these follow
\[
\Sigma_3 = k^{7/2} \left\{ \frac{L(1, \chi)}{\pi} - \frac{6L(3, \chi)}{\pi^3} \right\}.
\]
This gives
\[
\Sigma_3 \geq -\frac{6L(3, \chi)}{\pi^3}.
\]
If one could prove independently any appreciably better result, one could derive a sensational inequality for $L(1, \chi)$. For instance, if one could prove
\[
\Sigma_3 \geq -\frac{4}{\pi^3} \geq -\frac{5L(3, \chi)}{\pi^3},
\]
one could get by (6)
\[
L(1, \chi) > \frac{L(3, \chi)}{\pi^2}.
\]
Another possibility is that one can perhaps derive some connec-
tion between $\Sigma_1$ and $\Sigma_3$. For instance, if one could prove

$$\Sigma_3 \geq - k^2 \log k \Sigma_1,$$

then by (4) and (6), we could infer

$$L(1, \chi) > \frac{6L(3, \chi)}{\pi^2 (1 + \log k)}.$$

Even this would be a very worthwhile result, since the best known at present is, by Siegel's Theorem,

$$L(1, \chi) > \frac{L(3, \chi)}{k^\epsilon}$$

for $\epsilon > 0$ and large $k$.

**Bibliography**


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