AN INEQUALITY RELATED TO THE ISOPERIMETRIC INEQUALITY

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In this note we shall prove the following theorem.

**Theorem 1.** Let \( m \) be the measure of an open subset \( O \) of Euclidean \( n \)-space, and let \( m_1, \ldots, m_n \) be the \( (n-1) \)-dimensional measures of the projections of \( O \) on the coordinate hyperplanes. Then

\[
m^{n-1} \leq m_1m_2 \cdots m_n.
\]

Note that for \( n \)-dimensional intervals with faces parallel to the coordinate hyperplanes, (1) holds with the equality sign.

With any reasonable definition of the \( (n-1) \)-dimensional measure \( s \) of the boundary of \( O \), \( s \geq 2m_i \) for each \( i \), so that (1) gives

\[
m^{n-1} \leq s^n/2^n;
\]

this is the isoperimetric inequality, without the best constant. Since the proof of the isoperimetric inequality with the best constant is difficult,\(^1\) and since its applications do not necessarily require the best constant, our elementary proof of the theorem may be of some interest.

We first reduce the problem to a combinatorial one, in the following theorem.

**Theorem 2.** Let \( S \) be a set of cubes from a cubical subdivision of \( n \)-space; let \( S_i \) be the set of \( (n-1) \)-cubes obtained by projecting the cubes of \( S \) onto the \( i \)th coordinate hyperplane. Let \( N \) and \( N_i \) be the numbers of cubes in \( S \) and \( S_i \) respectively. Then

\[
N^{n-1} \leq N_1N_2 \cdots N_n.
\]

Assuming Theorem 2, we prove Theorem 1 as follows. Given \( \epsilon > 0 \), choose a cubical subdivision of \( n \)-space into cubes of side \( \delta \), with \( \delta \) so small that if \( S \) is the set of cubes interior to \( O \) forming the set \( \overline{S} \), \( \mu(O - \overline{S}) < \epsilon \) (\( \mu \) = measure). Then

\[
[\mu(\overline{S})]^{n-1} = N^{n-1}\delta^{n(n-1)} \leq (N_1\delta^{n-1}) \cdots (N_n\delta^{n-1}) \leq m_1 \cdots m_n,
\]

and since \( \epsilon \) is arbitrary, (1) follows.

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Proof of Theorem 2. If \( n = 2 \), the theorem is clear; we shall use induction on \( n \). Each cube of \( S \) projects into an interval on the first coordinate axis; let \( I_1, \ldots, I_k \) be the intervals thus obtained. Let \( T_i \) be the set of cubes projecting onto \( I_i \), and let \( T_{ij} \) be the set of \((n-1)\)-cubes obtained by projecting the cubes of \( T_i \) into the \( j \)th coordinate hyperplane \((j = 2, \ldots, n)\). Let \( a_i \) and \( a_{ij} \) be the numbers of cubes in \( T_i \) and \( T_{ij} \), respectively. Clearly

\[
\sum_{i=1}^{k} a_i = N, \quad a_i \leq N_1 \quad (i = 1, \ldots, k),
\]

\[
\sum_{i=1}^{k} a_{ij} = N_j \quad (j = 2, \ldots, n).
\]

Also, by induction,

\[
a_i^{n-2} \leq a_{i_1} \cdots a_{i_n} \quad (i = 1, \ldots, k).
\]

From (6) and the second part of (4) we obtain

\[
a_i^{n-1} \leq N_1 a_{i_2} \cdots a_{i_n} \quad (i = 1, \ldots, k).
\]

Now using successively the first part of (4), the above inequality, Hölder's inequality, and (5), we see that

\[
N = \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} N_1^{1/(n-1)} \prod_{j=2}^{n} a_{ij}^{1/(n-1)}
\]

\[
\leq N_1^{1/(n-1)} \prod_{j=2}^{k} \left( \sum_{i=1}^{m} a_{ij} \right)^{1/(n-1)} = \prod_{j=1}^{n} N_j^{1/(n-1)},
\]

as required.

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