462t. Werner Leutert: *On the convergence of approximate solutions of the heat equation to the exact solution.*

It is shown that an approximate solution of the heat equation can be obtained from a three line difference equation by using only half of the particular solutions of the form $e^{\alpha x}e^{\alpha t}$. The approximate solution will converge to the exact solution for all positive values of the mesh ratio $r = \Delta t/(\Delta x)^2$ and it will be stable in the sense that small changes in the initial condition vanish as the time $t$ is increased. von Neumann's test shows instability for all values of $r > 0$. (Received July 31, 1950.)

463t. Bertram Yood: *On fixed points for semi-groups of linear operators.*

Let $G$ be a semi-group of bounded linear operators on a normed linear space $X$, and $G^*$ be the family of adjoints of elements of $G$. Sets of conditions are given on $G$ which imply the existence of a nonzero fixed element for $G^*$ (in $X^*$). In particular if $X$ is the space of bounded functions on a set $S$, the results show, as a special case, the existence of a finitely-additive measure defined for all subsets of $S$ invariant under a solvable group of 1-1 transformations of $S$ onto $S$. This fact is due to von Neumann (Fund. Math. vol. 13 (1929)). (Received September 14, 1950.)

**Applied Mathematics**


The automata moves in an artificial environment having positions or states $q_i(i=1, \ldots, N_0)$. It has a repertory of moves that it can make, each called $m_{ij}$ ($j = 1, \ldots, N_i$). From state $q_i$ by move $m_{ij}$ it goes to a new uniquely determined state $q_k$, that is, $(q_i, m_{ij}) = q_k$. Each state $q_i$ is characterized by an aspect $a_i$ having the value $+1$ or $-1$. The $a_i$ is a "drive" in the psychological sense, and when $a_i$ is positive the automata is active. In state $q_i$ the automata initially randomly chooses an $m_{ij}$ where all the $m$'s have an equal probability. In the case $(q_i, m_{ij}) = q_{i+1}$ whose $a_{i+1}$ is negative (drive extinguished), then the probability is increased for choice $m_{ij}$ when in state $q_i$. In $(q_i, m_{ij})$ there is a transfer relation such that when some $m_{i+1, k}$ of $q_{i+1}$ has a probability greater than $2/N_i$, then the probability of taking $m_{ij}$ in $q_i$ is also increased. The automata as postulated can learn its way through a maze, learning from the goal backwards; it can remember the solution to two or more mazes; it forgets non-used information; and its behavior is not predictable. (Received September 5, 1950.)


Starting with the definition of stability in the case of linear varying-parameter systems: a system is stable if and only if every bounded input produces a bounded output, it is shown that the necessary and sufficient condition for stability is that the impulsive response of the system $W(t, \tau)$ should belong to $L(0, \infty)$ for all $t$ ($W(t, \tau)$ is the response at $t$ to a unit impulse applied at $t-\tau$). The system function of a linear varying-parameter system is related to $W(t, \tau)$ through $H(s; t) = \int_0^t W(t, \tau)e^{-st}d\tau$. From this it follows that the system function of a stable system is analytic in the right half and on the imaginary axis of the $s$-plane for all $t$. This result can be applied with advantage to the investigation of stability of linear varying-parameter systems. In particular, it yields useful criteria of stability for differential equations having periodic coefficients. (Received September 14, 1950.)
466t. L. A. Zadeh: Initial conditions in linear varying-parameter systems.

Consider a linear varying-parameter system $N$ whose behavior is described by an $n$th order linear differential equation $L(p; t)v(t) = u(t)$. Let $u(t)$ be zero for $t < 0$ and let the initial values of $v(t)$ and its derivatives be $v^{(0)}(0) = \alpha_v (v = 0, 1, \ldots, n-1)$. Let $H(s; t)$ be the system function of $N$. When the system is initially at rest (that is, all $\alpha_v$ are zero), the response of $N$ to $u(t)$ may be written as $v(t) = \int_0^t H(s; t) U(s) \, ds$ (see abstract 56-6-465). When, on the other hand, some of the $\alpha_v$ are not zero, the expression for the response to a given input $u(t)$ becomes $v(t) = \int_0^t H(s; t) [U(s) + A(s)] \, ds$, where $A(s)$ is a polynomial in $s$ and $p_0$ given by $A(s) = \frac{[L(s; 0) - Lp_0; 0]}{(s - p_0)^n}$ ($p_0$ represents a differential operator such that $p_0v^{(0)} = \alpha_v$). $\Delta(s)$ is essentially the Laplace transform of a linear combination of delta-functions of various order (up to $n-1$) such that the initial values of the derivatives of the response of $N$ to this combination are equal to $\alpha_v$. (Received September 14, 1950.)

**TOPOLOGY**


J. H. C. Whitehead has defined (Ann. of Math. vol. 42 (1941) pp. 409-428) a product which associates with elements $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(X)$. The authors show how to define three new products, as follows: (a) A product which associates with elements $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(X, A)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(X, A)$. (b) A product which associates with elements $\alpha \in \pi_p(A/B)$ and $\beta \in \pi_q(A \cap B)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(A/B)$. Here the sets $A$ and $B$ are a covering of the space $X = A \cup B$, and $\pi_p(A/B)$ is the $p$-dimensional homotopy group of this covering which has been introduced by the authors (Bull. Amer. Math. Soc. Abstract 56-3-208). (c) Let $(X; A, B)$ be a triad (see A. L. Blakers and W. S. Massey, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) p. 323), then there is a product which associates with elements of $\pi_p(A/B)$ and $\pi_q(X, A \cap B)$ an element of $\pi_{p+q-1}(X; A, B)$. The bilinearity of these three new products is established under suitable restrictions, and relationships between the various products are proved. The behavior of the products under homomorphisms induced by a continuous map or a homotopy boundary operator is also studied. (Received August 30, 1950.)

468t. A. L. Blakers and W. S. Massey: The triad homotopy groups in the critical dimension.

Let $X^* = X \cup \xi^1 \cup \xi^2 \cup \cdots \cup \xi^n$ be a space obtained by adjoining the $n$-dimensional ($n \geq 2$) cells $\xi^p$ to the connected, simply connected topological space $X$. Let $\xi^n = \xi^n_1 \cup \xi^n_2 \cup \cdots \cup \xi^n_m$ and $\xi^n = X \setminus \xi^n$. Assume that the space $\xi^n$ is arcwise connected, and that the relative homotopy groups $\pi_p(X, \xi^n)$ are trivial for $1 \leq p \leq m$, where $m \geq 1$. Then it is known that the triad homotopy groups $\pi_q(X^*; \xi^n, X)$ are trivial for $2 \leq q \leq m + n - 1$. The authors now show that under the assumption of suitable "smoothness" conditions on the pair $(X, \xi^n)$ (for example, both $X$ and $\xi^n$ are compact A.N.R.'s), there is a natural isomorphism of the tensor product $\pi_0(\xi^n, \xi^n) \otimes \pi_{m+n}(X/\xi^n)$ onto the triad homotopy group $\pi_{m+n}(X^*; \xi^n, X)$. This isomorphism is defined by means of a generalized Whitehead product. The Freudenthal "Ein­hängung" theorems in the critical dimensions can easily be derived from this theorem;