dromy theorem which is based on the Riemann mapping theorem. The selection of material from a rich field always involves a question of choice. An account of Riemann's classic work on the hypergeometric functions would have found a fitting niche in this chapter (cf. Ahlfors, *Complex analysis*) but its omission is certainly understandable.

The seventh chapter treats entire functions and meromorphic functions in the finite plane. The material treated includes the well-known expansion theorems of Weierstrass and Mittag-Leffler, growth questions, and the Picard theorems treated via the Bloch theorem. The remaining two chapters treat elliptic functions, the gamma and zeta functions, and Dirichlet series.

This brief account of the book indicates its scope and point of view. As we have remarked there is an abundance of exercises on which the good student may sharpen his mathematical teeth. He will have more than one occasion to test his skill with category arguments. On the other hand, the reader will note an absence of the treatment of the more delicate boundary problems which appeal either to a refined use of the topology of the plane or to methods involving Lebesgue integration. This is of course in accord with the stated program and intent of the book.

This book is a very welcome addition to the collection of texts on the theory of analytic functions which are now available in English. It will be a rewarding experience to the earnest student.

MAURICE HEINS

The stability of rotating liquid masses. By R. A. Lyttleton. Cambridge University Press, 1953. 10+150 pp. $6.50.

Ever since Newton deduced from his theory of gravitation that the shape of the earth must be an oblate spheroid, there has been intensive research into the question of the possible equilibrium shapes of rotating liquids. Maclaurin and Clairaut showed that for any value of angular momentum a spheroid is a possible equilibrium form. In 1834 Jacobi showed that if the angular momentum is greater than a certain amount an ellipsoid with three unequal axes is also a possible form of relative equilibrium.

The question of the stability of these equilibrium forms was first investigated by Poincaré in 1885. There are two different kinds of stability possible for rotating systems, known as "secular" and "ordinary" stability. To explain the distinction, consider a system rotating with constant angular velocity ω and assume it has n degrees of freedom $q=(q_1, q_2, \cdots, q_n)$ relative to a set of axes rotating with angular velocity ω.

In rotating systems the usual potential energy V must be replaced by the quantity $V-\omega^2I/2$ where I is the moment of inertia of the system. The equilibrium conditions are

$$\frac{\partial}{\partial q_n}\left(V - \frac{1}{2}\omega^2 I\right) = 0.$$

For small displacements from equilibrium we may assume that

$$V - \frac{1}{2}\omega^2 I = (q, Bq)$$

where B is a real symmetric matrix and the parenthesis denotes the usual real scalar product. We shall assume also that the kinetic energy relative to the rotating axis is

$$T = \frac{1}{2}(\dot{q}, A\dot{q})$$

where the dots denote differentiation with respect to the time t and A is a real symmetric, positive definite matrix. If there are no external forces and if friction is neglected, then the equations of motion will be

$$A\ddot{q} + \omega G\dot{q} + Bq = 0 \tag{1}$$

where G is a real skew-symmetric matrix. The term $\omega G\dot{q}$ is the so-called gyroscopic term. Its presence is due to the fact that the equations of motion are written in a rotating, and not a static, set of coordinate axes.

The free oscillations of the system can be found by substituting $q=e^{\lambda t}q_0$ in equation (1). We get

$$(\lambda^2 A + \omega\lambda G + B)q_0 = 0. \tag{2}$$

This will have a nontrivial solution if

$$\det(\lambda^2 A + \omega\lambda G + B) = 0. \tag{3}$$

Since the transpose of $\lambda^2 A+\omega\lambda G+B$ is $\lambda^2 A-\omega\lambda G+B$, and since the determinant of a matrix and its transpose are equal, it follows that $-\lambda$ is a root of (3) if λ is; consequently (3) is an polynomial equation in λ^2.

The rotating system is said to be ordinarily stable if all the values of λ^2 which satisfy (3) are real and negative. In that case, all free oscillations are bounded and the system oscillates in the neighborhood of equilibrium in response to any disturbance.

Write (2) as follows:

$$(4) \qquad (\lambda A + \omega G + B\lambda^{-1})q_0 = 0.$$

Since A, B, and G are matrices with real elements, the complex conjugate of this set of equations is

$$(5) \qquad (\bar{\lambda} A + \omega G + B\bar{\lambda}^{-1})\bar{q}_0 = 0.$$

Add the scalar product of (4) by $\bar{q}_0$ to the scalar product of (5) by q_0. We get

$$(6) \qquad (\lambda + \bar{\lambda})(\bar{q}_0, A\bar{q}_0) + (\lambda + \bar{\lambda}) \, |\, \lambda \,|^{-2}(\bar{q}_0, Bq_0) = 0$$

because $(\bar{q}_0, Gq_0) = -(q_0, G\bar{q}_0)$.

A rotating system for which B is positive-definite is said to be secularly stable. In this case (6) shows that $\lambda + \bar{\lambda}$ must be zero, that is, λ is purely imaginary; consequently, a system which is secularly stable is also ordinarily stable. On the other hand, if the system is secularly unstable, that is, if B is not positive-definite, then equation (6) may be satisfied by putting

$$|\, \lambda \,|^2 = -\, (\bar{q}_0, Bq_0)/(\bar{q}_0, Aq_0).$$

If these values of λ are also purely imaginary, the system will still be ordinarily stable; consequently, a system may be secularly unstable but ordinarily stable.

The difference between secular and ordinary stability becomes evident when friction is taken into account. In that case the equations of motion become

$$A\ddot{q} + (F + \omega G)\dot{q} + Bq = 0$$

where F is a real, positive-definite matrix. If we put $q = e^{\lambda t}p$, then just as before we can show that

$$(7) \qquad [\lambda + \bar{\lambda}]\,[(p, Ap) + |\, \lambda \,|^{-2}(\bar{p}, Bp)] + (\bar{p}, Fp) = 0.$$

If B is positive-definite, then this equation implies that $\lambda + \bar{\lambda}$ is negative, that is, the real part of λ is negative; consequently, if the system is secularly stable, friction will dampen all disturbances from equilibrium.

If B is not positive-definite, the bracket in (7) will become negative and then the real part of λ will be positive and its magnitude will depend on the strength of the frictional forces; consequently, when a rotating system is ordinarily stable, frictional forces may increase the disturbances from equilibrium and so the system will not stay near equilibrium. Since this displacement from equilibrium depends upon the strength of the frictional forces, the system may stay near equilibrium for an extremely long time. For example, the lunar orbit is ordinarily stable but secularly unstable and its change from equilibrium under the influence of frictional forces is extremely slow.

Poincaré showed that, as the angular momentum increased, the MacLaurin spheroids which were originally secularly stable become secularly and ordinarily unstable at that value of angular momentum for which Jacobi proved the existence of a rotating ellipsoid. Using ellipsoidal harmonics, Poincaré showed further that the Jacobi ellipsoids are secularly stable until a certain value of angular momentum at which point there exist certain pear-shaped figures of equilibrium.

Darwin suggested that as the angular momentum increased the furrow in these pear-shaped figures might deepen until the liquid split in two independent bodies. This fission process could be the origin of binary stars, planets, and satellites. It was proved by Jeans, Liapounoff, and E. Cartan that these pear-shaped figures were secularly unstable.

The crux of the problem, however, is whether these figures are ordinarily stable. Because of the small frictional forces involved in astronomical phenomena, ordinary stability would be sufficient to permit the formation of binary systems by the above method. E. Cartan eventually proved that these figures are ordinarily unstable.

The book under review presents the mathematical details of the theory outlined above and also discusses their cosmological implications. The account starts with a discussion of the stability of rotating system and follows with a discussion of the spheroidal and ellipsoidal figures of equilibrium. Next, the theory of ellipsoidal harmonics and Lamé functions is developed. This theory is used to present Poincaré's and Cartan's discussion of the stability of the equilibrium figures. Finally, the author shows that the theory of binary fission advocated by Darwin and Jeans is untenable.

The author is to be commended for preparing this clear, logical account of a subject which has excited the interest of the greatest mathematicians.

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