BOOK REVIEWS

Functionals of finite Riemann surfaces. By M. M. Schiffer and D. C. Spencer. Princeton University Press, 1954. 10+451 pp. $8.00.

In the terminology of the authors a finite Riemann surface is a Riemann surface of finite genus with a finite number of nondegenerate boundary components. Such a surface has a double, obtained by reflection across the boundary, and one of the main features of the book is the systematic use of this symmetrization process. The authors even introduce a disconnected double of a closed Riemann surface, and for nonorientable Riemann surfaces the double is a two-sheeted orientable covering surface.

The first three chapters contain a development of the classical theory. The topological classification is taken for granted, but the existence theorems are carefully proved by the method of orthogonal projection in de Rham's and Kodaira's version. The treatment of Abelian differentials includes the Riemann-Roch theorem. A somewhat disturbing omission in this introductory part is the lack of practically all references to covering surfaces, in spite of their use later on.

The more advanced theory begins with the fourth chapter. The main interest, throughout the book, is attached to the so-called *domain functionals*, represented by Green's and Neumann's function, harmonic measure, the period matrix, and similar quantities. The first task is to compare different functionals of the same domain, and this is accomplished by deriving all functionals from a common source. The simplest way is to start from normalized integrals of the third kind.

We introduce some of the notations used by the authors. For a closed surface $\mathcal{J}$, let $\Omega_{qq_0}(p)$ be the integral with logarithmic poles at q, q_0 and single-valued real part. Then

$$\frac{\partial^2 \Omega_{qq_0}(p)}{\partial p \partial q} = -\frac{1}{[z(p) - z(q)]^2} + \text{regular terms}$$

is a bilinear differential on $\mathcal{J}$ (independent of the auxiliary point q_0). Suppose now that $\mathcal{J}$ is the double of $\mathfrak{M}$, and let $\tilde{q}$ denote the symmetric point of q. The authors introduce

$$L(p, q) = -\frac{1}{\pi} \frac{\partial^2 \Omega_{q\tilde{q}}(p)}{\partial p \partial q}$$

and prove on one hand the symmetry relations

$$L(p, q) = L(q, p), \qquad L(p, \tilde{q}) = (L(q, \tilde{p}))^{-},$$

on the other hand the reproduction formulas

$$(df, L(p, \tilde{q})dz) = -f'(q), \qquad (df, L(p, q)dz) = 0.$$

Here $df \in M$, where M is the Hilbert space of square integrable differentials on $\mathfrak{M}$, and the inner products are extended over $\mathfrak{M}$. These formulas show that $-L(p, \tilde{q})$ is the Bergman kernel function for the space M. The Green's and Neumann's functions are expressed by

$$G(p, q) = \tfrac{1}{2}\{\Omega_{q\tilde{q}}(p) - \Omega_{q\tilde{q}}(\tilde{p})\},$$

$$N(p, q, q_0) = \tfrac{1}{2}\{\Omega_{qq_0}(p) + \Omega_{\tilde{q}\tilde{q}_0}(\tilde{p}) + \Omega_{qq_0}(\tilde{p}) + \Omega_{\tilde{q}\tilde{q}_0}(p)\}.$$

Finally, the Abelian differentials of the first kind that correspond to a Betti basis $\{K_\mu\}$ can be defined as

$$Z_\mu'(q) = -\frac{2}{\pi}\int_{K_\mu} \frac{\partial^2 G(p, q)}{\partial p \partial q}\, dp.$$

This definition implies

$$(dw, dZ_\mu) = -\int_{K_\mu} dw.$$

The next chapter is concerned with the situation that arises when a surface $\mathfrak{M}$ is imbedded in another surface $\mathfrak{R}$. Let the corresponding kernels be $L(p, q)$ and $\mathfrak{L}(p, q)$ respectively. The difference

$$l(p, q) = L(p, q) - \mathfrak{L}(p, q)$$

is then regular on $\mathfrak{M}$ and can be shown to satisfy a number of important identities. It is easy to generalize to the case of an $\mathfrak{M}$ which is not necessarily connected.

The consideration of $l(p, q)$ leads not only to identities, but also to inequalities which express that certain quadratic forms are positive definite. These inequalities characterize the imbedding so completely that it becomes possible to decide whether a locally given bilinear differential can or cannot be extended to all of $\mathfrak{M}$. With the help of this result the authors derive a very strong theorem which concerns the possibility of imbedding one Riemann surface in another, in terms that generalize the well-known schlichtness conditions of Grunsky. It is regrettable that the otherwise lucid reasoning becomes quite obscure at this crucial point. Certainly, the reader expects to be told, in no uncertain terms, how the imbedding is constructed. Instead, the authors merely hint that a certain relation may be used to extend the mapping "by continuation." It seems likely that a proof can be carried out along such lines, but the impression remains that an essential and perhaps difficult part of the argument has been omitted.

The theory of subdomains is continued in the sixth chapter which by its novelty and depth is one of the most fascinating sections of the book. Let M and R be the spaces of square integrable differentials over $\mathfrak{M}$ and $\mathfrak{R}$, $\mathfrak{M}\subset\mathfrak{R}$. Given $f'\in M$, the inner product $(g', f')_{\mathfrak{M}}$ defines a linear function of $g'\in\mathfrak{R}$, and consequently there exists a $Tf'\in\mathfrak{R}$ with the property that

$$(g', f')_{\mathfrak{M}} = (g', Tf')_{\mathfrak{R}},$$

for all $g'\in\mathfrak{R}$. The linear mapping T is considered as a canonical mapping of M into R.

Actually, the authors proceed somewhat differently and define directly two linear operators.

$$T_f(q) = (\mathcal{L}(q, p)dp, f'dp)_{\mathfrak{M}}, \qquad \tilde{T}_f(q) = (f'dp, \mathcal{L}(p, \bar{q})dp)_{\mathfrak{M}}.$$

Here $\tilde{T}_f$ corresponds to the operator that we described above, while T_f is a generalized Hilbert transform which has to be evaluated as a principal value. More generally, T and $\tilde{T}$ may be defined for differentials that are analytic in $\mathfrak{M}$ and $\mathfrak{R}-\mathfrak{M}$ but may be discontinuous on the boundary of $\mathfrak{M}$. Then $\tilde{T}$ is an isometry, and $T_{f_1}\perp\tilde{T}_{f_2}$ with respect to $\mathfrak{R}$. The algebra generated by T and $\tilde{T}$ satisfies the relations

$$T^2 = I + \tilde{T}, \qquad \tilde{T}^2 = -\tilde{T}, \qquad T\tilde{T} = \tilde{T}T = 0.$$

The first of these is far from evident.

For many purposes it is sufficient to consider operations in the domain $\mathfrak{M}$ only. It is proved that the corresponding restrictions t, $\tilde{t}$ of T, $\tilde{T}$ are completely continuous. Thus one obtains an interesting spectral theory which yields powerful generalizations of the methods of Poincaré and Fredholm in classical potential theory.

The last part of the book, except for an appendix on higher dimensional Kähler manifolds, is devoted to variational techniques. Here again, the main problem is to find the variations of the domain functionals. The technique, which goes back to Hadamard and has been perfected by Schiffer, is described in great detail, and different types of variations are introduced which involve cutting holes, attaching handles, cross-caps and disks. The connections with quadratic differentials are explained and utilized.

An important problem, which has hardly been touched before, is the construction of variations which preserve the conformal type. Once more, the reader will find that the authors are reluctant to detail proofs that cannot be readily expressed through analytic formulas. For instance, they use the natural device of introducing, as conformal modules, the real periods of a differential dZ (with given imaginary periods) and the integrals

$$\int_{p_1}^{p_2} dZ$$

between the zeros p_2 of Z'. In the situation that arises through a variation they assert, as a triviality, that these quantities determine the Riemann surface. The reviewer agrees that a precise proof is not difficult, but it will necessarily involve considerations that the text does not even touch upon. Apart from such minor inconsistencies the proofs seem satisfactory and the results are very far-reaching.

Numerous applications are given, and it is shown, in particular, that the period matrix depends differentiably on the moduli. There is no attempt, however, to introduce a complex structure on the space of Riemann surfaces.

The last chapter is rather loosely connected with the rest of the book. It gives a concise and readable presentation of the Hodge theory of harmonic differentials on Kählerian manifolds, together with its extension to manifolds with boundary.

The authors must be congratulated on having got out of their hands and before the eyes of the mathematical public a volume that must have been very difficult to edit. It contains a plethora of ideas, each interesting in its own right, and on the whole they have been tied together in a successful manner. At a first reading the wealth of formulas is almost forbidding, especially since the authors have not been very fortunate in their choice of notations. However, the patient reader will be richly rewarded and will become aware of many challenging problems that remain to be solved.

The publishers have gone out of their way to give the formulas an attractive appearance, and the proofreading is excellent.

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Topological dynamics. By W. H. Gottschalk and G. A. Hedlund. American Mathematical Society Colloquium Publications, vol. 36. Providence, American Mathematical Society, 1955. 8+148 pp. $5.10.

The authors begin by explaining that by *topological dynamics* they mean "the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics." Thus, they say, *topological* has reference to the mathemati-