RESEARCH PROBLEMS


Let \( f(x) \) be a monotone increasing function of \( x \) with positive continuous derivative for \( x \geq 0 \), with \( f(0) = 0, f(\infty) = \infty \). Consider the equation

\[
(1) \quad f(x) = y,
\]

possessing the unique solution \( x = f^{-1}(y) \) for \( y \geq 0 \). Let

\[
(2) \quad x_{n+1} = x_n + \frac{y - f(x_n)}{f'(x_n)}, \quad x_0 = x,
\]

be the sequence of successive approximations to \( f^{-1}(y) \) furnished by Newton's method.

Determine \( z^* = z(a, b, n) \) so that

\[
(3) \quad \max_{x_n \in [a, b]} |x_n - f^{-1}(y)|
\]

is a minimum, where \( 0 < a < b < \infty \), and determine the asymptotic behavior of \( z(a, b, n) \) as \( n \to \infty \).

For \( f(x) = x^3 \), it is known that \( z(a, b, n) \to (ab)^{1/4} \) as \( n \to \infty \). (Received November 26, 1956.)


At the present time, there is no systematic technique for solving the problem of maximizing the linear form

\[
L(x) = \sum_{i=1}^{M} a_i x_i
\]

subject to the constraints \( \sum_{i=1}^{M} b_{ij} x_i \leq c_i, \; i = 1, 2, \ldots, M \), where the \( a_i \) and \( b_{ij} \) are positive integers, or zero, and the \( x_i \) are constrained to be positive integers or zero. On the other hand, if this constraint on integral solutions is removed, the solution is readily obtained for small \( M \), and there exist effective algorithms for large \( M \).

For the case \( M = 1 \), let \( f_N(c) \) denote the maximum of \( L(x) \) under integral constraints and \( g_N(c) \) denote the solution under the constraint \( x_i \geq 0 \). Define the function

\[
\phi(N) = \sup_{a_i > 0} \left[ \sup_{c_i \geq 0} \frac{g_N(c)}{f_N(c)} \right].
\]

What is the order of magnitude of \( \phi(N) \) as \( N \to \infty \), and in particular, is it bounded?

Consider the corresponding problem for general \( M \) where

\[
\phi_M(N) = \sup_{a_i > 0} \left[ \sup_{c_i \geq 0} \frac{g_N(c_1, c_2, \ldots, c_M)}{f_N(c_1, c_2, \ldots, c_M)} \right]
\]

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