RESEARCH PROBLEMS


(1) Let \( 0 < A < 1 \). Let \( B \) be the set of all positive integers \( n \) such that there exist \( n \) positive numbers \( a_1, a_2, \ldots, a_n \) such that the polynomial

\[
(x^2 - 2Ax + 1) \prod_{i=1}^n (x + a_i)
\]

has all non-negative coefficients. It is known that \( B \) is nonempty. (See P. M. Lewis, The concept of the one in voltage transfer synthesis, IRE Trans. Vol. CT-2, pp. 316-319, December, 1955.) Find the least element of \( B \).

(2) Let \( 0 < A < 1 \) and let \( N \) be the smallest integer in \( B \), as described in (1) above. Does there exist \( b > 0 \) such that

\[
(x^2 - 2Ax + 1)(x + b)^N
\]

has all non-negative coefficients?

(3) Generalize the questions raised in (1) and (2) above to the case where the factor \( x^2 - 2Ax + 1 \) is replaced by an arbitrary real polynomial, say \( \sum_{i=0}^n C_i x^i \), having no positive real roots. (Received September 13, 1957.)

2. Louis Weinberg: Decomposition of Hurwitz polynomials.

Let \( q(s) = \sum_{k=0}^n a_k s^k \) represent a Hurwitz polynomial with real coefficients, i.e., all of its zeros have negative real parts. Can \( q(s) \) be divided into the arithmetic sum of two polynomials,

\[
q(s) = q_1(s) + q_2(s)
\]

each of which has positive coefficients and only nonpositive real roots? This can easily be done in particular cases; for example, if \( q(s) = (s^2 + 2s + 5)(s + 4) = s^4 + 6s^3 + 13s + 20 \), then \( q_1(s) = s^2 + 6s + 11s + 6 = (s + 1)(s + 2)(s + 3) \) and \( q_2(s) = 2s + 14 \). If this can be shown to be impossible in the general case, can the decomposition always be carried out with three polynomials,

\[
q(s) = q_1(s) + q_2(s) + q_3(s)
\]
each of which again has positive coefficients and only nonpositive real roots? (Received September 19, 1957.)

3. R. E. Bellman: Number theory. I.

The problem of generating the integer solutions of the equation \( x^2 + y^2 = 1 \) (mod \( p \)) by means of the formula \( x_n = \cos n\theta, y_n = \sin n\theta \), where \( (x_1, y_1) \) is a fundamental solution which we can write symbolically in the form \( x_1 = \cos \theta_1, y_1 = \sin \theta_1 \), has been extensively studied. What are the corresponding results for the equations \( x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \) (mod \( p \))?

In particular, for the equation \( x_1^2 + x_2^2 + x_3^2 = 1 \) (mod \( p \)), what subset of solutions do we obtain by means of the formulas

\[
\begin{align*}
    x_1 &= \cos k\theta_1 \cos l\theta_2, \\
    x_2 &= \cos k\theta_1 \sin l\theta_2, \\
    x_3 &= \sin k\theta_1,
\end{align*}
\]

where \( k, l = 0, 1, \ldots \), and \( \theta_1, \theta_2 \) correspond to certain primitive solutions? (Received May 22, 1957.)
4. R. E. Bellman: *Number theory. II.*

Consider the same type of problem for the multiplicative form

\[ x^3 + y^3 + z^3 - 3xyz \]

and for the circulant functions of higher order. (Received May 22, 1957.)

5. R. E. Bellman: *Number theory. III.*

Consider the equation \( y^2 = 4x^3 - g_x - g_z \) which may be uniformized by means of the Weierstrassian elliptic functions \( x = p(u), y = p'(u) \). What subset of solutions of the congruence \( y^2 = 4x^3 - g_x - g_z \) (mod \( p \)) can be obtained by means of the formulas \( x = p(mu + nv), y = p'(mu + nv), m, n = 0, 1, 2, \ldots \), (not both zero simultaneously), where \( u \) and \( v \) correspond to certain primitive solutions?

Consider the similar problem for \( y^2 = (1 - x^3)(1 - k^2x^3) \) which can be uniformized by means of Jacobian elliptic functions. (Received May 22, 1957.)

6. R. E. Bellman: *Number theory. 1.*

Let \( x \) be an irrational number in \([0, 1]\) and let \( g(y; a, b), 0 \leq a < b \leq 1 \) be a periodic function of \( y \) with period 1 defined by the conditions \( g(y; a, b) = 1, a \leq y \leq b, g(y; a, b) = 0 \) elsewhere for \( y \) in \([0, 1]\). Define the function

\[ f_N(z, x) = g(x; a, b) + g(x + y; a, b) + \cdots + g(x + Nz; a, b) \]

for \( 1 \geq z \geq 0 \), equal to the number of elements of the finite sequence \( \{nx + z\} \), \( n = 0, 1, \ldots, N \), falling inside \([a, b]\), modulo one.

The Weyl equidistribution theorem asserts that \( f_N(z, x)/(N+1) \sim b - a \) as \( N \to \infty \). It is easy to show via Fourier series that

\[ \int_0^1 \int_0^1 \frac{\left[f_N(z, x) - (N + 1)(b - a)\right]^2dx}{(N + 1)^{1/2}} \sim (N + 1)c_0(a, b) \]

as \( N \to \infty \).

This suggests that the quantity

\[ u_N(z, x) = \frac{f_N(z, x) - (N + 1)(b - a)}{(N + 1)^{1/2}} \]

possesses asymptotic moments of all orders.

Does

\[ \lim_{N \to \infty} \int_0^1 \int_0^1 \left(\frac{f_N(z, x) - (N + 1)(b - a)}{(N + 1)^{1/2}}\right)^k dx \sim (N + 1)c_0(a, b) \]

\( k = 2, 3, \ldots \) exist, and if so what is the limiting distribution? (Received July 15, 1957.)

7. R. E. Bellman: *Number theory. 2.*

Consider the same problem for the function

\[ f_N(z, y, x) = g(x; a, b) + g(x + 2y + z; a, b) + \cdots + g(x + 2Ny + N^2z; a, b) \]

with \( x \) irrational, \( y \) and \( z \) in \([0, 1]\). As above, it is easy to show via Fourier series that

\[ \int_0^1 \int_0^1 \int_0^1 \left[f_N(z, y, x) - (N + 1)(b - a)\right]^2dxdydz \sim (N + 1)c_0(a, b) \]

as \( N \to \infty \).
Does
\[ \lim_{N \to \infty} \int_0^1 \int_0^1 \int_0^1 \left( \frac{f(x, y, z) - (N + 1)(b - a)}{(N + 1)^{1/2}} \right)^k \, dx \, dy \]
exist for \( k = 1, 2, \ldots \), and if so, what is its value?

There are corresponding versions of this problem for polynomials of all orders.
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Consider the second order linear differential equation \( u'' + (1 + \lambda g(x))u = 0 \), where \( \lambda \) is a real constant and \( \int_0^\infty |g(x)| \, dx < \infty \). Let \( u_1(x) \) be the solution specified by \( u_1(0) = 0 \), \( u_1'(0) = 1 \). It is known that \( u \sim r(\lambda) \sin (x + \theta(\lambda)) \) as \( x \to \infty \).

Taking \( \lambda \) to be complex variable, what are the analytic properties of the functions \( r(\lambda) \) and \( \theta(\lambda) \)? In particular, where are the singularities nearest the origin?

If \( g(x) > 0 \) for \( x \geq 0 \), is the singularity nearest the origin on the negative axis?
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