ON ISOMORPHISMS OF GROUP ALGEBRAS

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With every locally compact topological group $G$ there is associated its group algebra $L(G)$, the space of all complex Haar-integrable functions on $G$ with convolution as multiplication. Considerable work has been done toward discovering the extent to which the algebraic structure of $L(G)$ determines $G$ (see [1; 2; 5]), but some very specific questions have been left unanswered. For instance: Is the group algebra of the circle isomorphic to that of the torus? The theorem announced here stems from this question.

**Theorem.** The group algebra of a locally compact topological group $T$ is isomorphic to that of the circle group $C$ if and only if $T$ is a direct sum $C+F$, where $F$ is a finite abelian group.

The proof leans heavily on that of Theorem 1 of [4]. In the outline below we will mainly be concerned with pointing out the changes in [4] which are needed to yield the stated result.

If $L(T)$ and $L(C)$ are isomorphic, then $T$ is abelian, and the dual group $\Gamma$ of $T$ is homeomorphic to $\mathbb{Z}$, the group of all integers (the dual group of $C$) [2, p. 478]. Thus $\Gamma$ is discrete and countable, and $T$ is a compact abelian group with countable base.

Abelian groups will be written additively; for $x \in T$ and $\phi \in \Gamma$ the symbol $(x, \phi)$ will stand for the value of the character $\phi$ at the point $x$; the Haar measure on $T$ will be denoted by $m$.

**Lemma 1.** Corresponding to every $E \subset T$ with $m(E) > 0$, there is only a finite set of characters $\phi$ such that, for all $x \in E$,

$$1 - (x, \phi) < 1.$$  

Note that (1) holds if and only if the real part of $(x, \phi)$ exceeds $1/2$. If $f$ is the characteristic function of $E$ and if $\phi$ satisfies (1), then $\left| \int_T (x, \phi)f(x)dx \right| > m(E)/2$, and the lemma follows from the Bessel inequality.

**Lemma 2.** Every infinite subset $A$ of $\Gamma$ contains an infinite subset $B$, such that for some $x \in T$ the inequality

$$\left| 1 - (x, \phi) \right| \geq 1$$

holds for every $\phi \in B$.

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This is proved by repeated application of Lemma 1.

If now $\psi$ is an isomorphism of $L(T)$ onto $L(C)$, $\psi$ can be extended to an isomorphism of the measure algebras $M(T)$ and $M(C)$, and [2, p. 479] there is a one-to-one mapping $\alpha$ of $J$ onto $\Gamma$ such that the Fourier-Stieltjes coefficients of $\psi(\mu)$ are

$$c_n(\psi(\mu)) = \int_T (-x, \alpha(n)) d\mu(x) \quad (n \in J, \mu \in M(T)).$$

For $x \in T$, let $e_x$ be the measure of mass 1 which is concentrated at $x$, and put $\mu_x = \psi(e_x)$. Then $c_n(\mu_x) = (-x, \alpha(n))$, and

$$\mu_x \ast \mu_y = \mu_{x+y} \quad (x, y \in T).$$

The mapping $x \mapsto \mu_x$ is thus an isomorphism of $T$ into $M(C)$.

The discrete parts $\lambda_x$ of $\mu_x$ also satisfy (4), and there is a mapping $\beta$ of $J$ into $\Gamma$ such that

$$c_n(\lambda_x) = (-x, \beta(n)) \quad (n \in J, x \in T);$$

the lemma used in Step 5 of [4] must here be applied to $C \times T$ in place of $C \times C$. Since $\lambda_x$ is discrete, $c_n(\lambda_x)$ is an almost periodic function on $J$, for each $x \in T$. Arguing as in Step 6 of [4], we find that there is a positive integer $k$ and a set $E \subset T$ with $m(E) > 0$, such that

$$|1 - (x, b(n))| < 1 \quad (n \in J, x \in E),$$

where $b(n) = \beta(n+k) - \beta(n)$. By Lemma 1, the sequence $\{b(n)\}$ has only a finite set of values, so that the almost periodicity of $\{(x, b(n))\}$ implies that $\{(x, b(n))\}$ is actually periodic, for every $x \in T$. A compactness argument now shows that $\{b(n)\}$ is itself periodic, with period $p$, say. If $q = kp$, it follows that

$$\beta(n + q) + \beta(n - q) = 2\beta(n) \quad (n \in J).$$

Next we put $\tau_x = (\lambda_x - \mu_x) \ast \lambda_{-x}$, so that

$$c_n(\tau_x) = 1 - (x, \gamma(n)) \quad (n \in J, x \in T),$$

where $\gamma(n) = \beta(n) - \alpha(n)$. Since the measures $\tau_x$ are continuous,

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{-N}^{N} c_n(\tau_x) = 0 \quad (x \in T).$$

These averages are uniformly bounded on $T$, so that (9) may be integrated; combined with (8), this implies that $\gamma(n) = 0$ except possibly on a set $S \subset J$ of density 0.

Thus if $S$ is infinite, $S$ contains an infinite set $\{n_k\}$ such that none
of the integers \( n_k + 1, n_k + 2, \cdots, n_k + k \) belong to \( S \), and by Lemma 2 there is an \( x \in T \) and a subsequence of \( \{ n_k \} \), again denoted by \( \{ n_k \} \), such that \( |c_{n_k}(\tau_x)| \geq 1 \). A subsequence of the measures

\[
d\sigma_k(\theta) = e^{-in_k\theta}d\tau_x(\theta)
\]

then converges weakly to a singular measure \( \sigma \) [3, p. 236] with \( |c_0(\sigma)| \geq 1 \) but \( c_n(\sigma) = 0 \) for all \( n > 0 \). This is impossible, so that \( S \) is finite.

It follows that \( \alpha = \pi \beta \), where \( \beta \) satisfies (7) and maps \( J \) onto \( \Gamma \), and \( \pi \) is a permutation of \( \Gamma \) which moves only a finite number of terms; \( \beta \) maps each residue class mod \( q \) onto an arithmetic progression in \( \Gamma \); hence \( \Gamma \) is finitely generated and is therefore a direct sum of a finite set of cyclic groups; since \( \Gamma \) is the union of a finite set of arithmetic progressions, only one of the direct summands can be infinite, so that \( \Gamma \) is a direct sum of \( J \) and a finite abelian group \( F \).

This proves one half of the theorem. The converse may be proved by defining

\[
\alpha(nq + k) = (n, f_k) \quad (n \in J, 1 \leq k \leq q),
\]

where \( f_1, \cdots, f_q \) are the elements of \( F \); it is easily verified that this induces, via (3), an isomorphism of \( L(T) \) onto \( L(C) \). In fact, every \( \alpha \) of the above form \( \alpha = \pi \beta \) has this property, as can be seen by an argument analogous to that on p. 50 of [4].

References