which is not covered by the general theorem follows. Quasi linear systems and complete families of periodic solutions are also included in this chapter.

The last few chapters are a very fine exposition of two-dimensional systems of differential equations. Chapter IX starts with an investigation of simple critical points. The index of simple critical points is computed. The chapter ends with a study of the Technique by which a differential equation can be extended from the Euclidean Plane to the Projective Plane, a method which has been neglected since Poincaré. Chapter X investigates general critical points. It is shown that for analytic systems the index of a critical point is equal to $1+1/2$ (number of elliptic section—number of hyperbolic section). The notion of the limit sets of a trajectory is taken up next. It is shown that the limit sets of a trajectory fall into four mutually exclusive categories. Critical points with a single zero characteristic root and structural stability are other highlights of this chapter.

Chapter XI discusses the equation $d^2x/dt^2+f(x)dx/dt+g(x)=e(t)$. We are concerned here with proving the existence and uniqueness of periodic solutions. The author exhibits a variety of techniques for accomplishing this goal, all taken from recent literature. Chapter XII studies the perturbation theory of second order differential equations. Sufficient conditions for the existence and uniqueness of periodic solutions of the perturbed systems are found. The stability of these periodic solutions is investigated. Other topics discussed in this chapter are the Stroboscopic method of Minorski, relaxation oscillation, and the stability zones of the Mathieu Equation.

Summarizing, this is a very interesting book containing a wealth of material. This work should be useful to a variety of mathematicians and physicists.

FELIX HAAS

Einführung in die transzendenten Zahlen. By. Th. Schneider. Berlin-Göttingen-Heidelberg, Springer, 1957. 7+150 pp. DM 21.60. Bound DM 24.80.

A complex number α is said to be algebraic or transcendental (over the rational numbers) according as it is or is not a root of an equation of the form $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$, where $a_0, \cdots, a_n$ are rational numbers. The distinction between these two kinds of numbers was recognized at least as early as Euler (1744), who asserted that the logarithm to a rational base of a rational number must be either rational or transcendental. This was a bold conjecture indeed, since at that time no example of a transcendental number was known, and no method was available even of constructing "artificial" transcendental numbers, much less of dealing with the question of the transcendence of a number given beforehand. It was a full century later that J. Liouville found a useful characterization of a class of transcendental numbers, by means of which such numbers could be constructed. He proved that if α is algebraic of degree n (that is, satisfies an irreducible polynomial equation of degree n), then there is a constant $c>0$ such that the inequality

$$(1) \qquad |\alpha - p/q| < c/q^{\mu}$$

has no solutions in integers p and q, if $\mu = n$. Hence α is transcendental if for fixed c and for every positive number μ the inequality (1) has a solution with $q>1$, and it is not difficult to see that the number $\alpha = \sum_1^\infty 2^{-k!}$, for example, has this property.

After Liouville there were two other notable results in the nineteenth century: in 1873 C. Hermite proved that e is transcendental, and nine years later F. Lindemann generalized Hermite's theorem in such a way that the transcendence of π became apparent from the relation $e^{\pi i} = -1$. However, even as recently as 1930 it would have been accurate to say that there was no real theory of transcendental numbers, but only a scattered set of results. Since that time, matters have improved considerably, and the present book demonstrates conclusively that there is now a respectable battery of general methods available both for unifying older work and for attacking new problems.

For the creation and development of these new methods, we are indebted primarily to four mathematicians: A. O. Gel'fond, K. Mahler, T. Schneider and C. L. Siegel. It is apparent on every page of Schneider's book that this is the work of a master of his field. It is evident too that the proofs have been reworked, simplified, and polished until they are as nearly elegant as this kind of estimative mathematics can be made. This has its advantages and disadvantages, of course; for one already familiar with the details of arguments similar to those omitted, the book is a pleasure to read, but to the beginner the gaps may be troublesome to fill. One might also wish that the author had included a final chapter on isolated results not fitting into the broad pattern of the book, and simultaneously completed the bibliography, but as it stands the book is a substantial contribution to the literature of the subject. (There are only two other modern books on transcendental numbers, by Siegel and Gel'fond; neither is as comprehensive as Schneider's.)

The first chapter is devoted to generalizations and improvements of Liouville's theorem. After intermediate work by A. Thue, Siegel, and F. J. Dyson, it has recently been proved by K. F. Roth that the theorem surrounding inequality (1) above remains true if μ is any constant larger than 2. The author combines Roth's method with earlier work by himself and Mahler to prove the following theorem, which reduces to Roth's in case $b=\eta=1$: *Let α be an irrational algebraic number, and let b be a positive integer. Let $\{p_\nu/q_\nu\}$ be an infinite sequence of rational numbers with $q_{\nu+1}>q_\nu>0$, where the denominators q_ν can be represented as products $q_\nu=q'_\nu \cdot q''_\nu$ of integers such that q''_ν is a nonnegative integral power of b. Put*

$$\eta = \limsup_{\nu\to\infty} \frac{\log q'_\nu}{\log q_\nu} \cdot$$

Then if $\mu>\eta+1$, the inequality (1) *has only finitely many solutions p/q from the sequence $\{p_\nu/q_\nu\}$.* (It might be worth mentioning that Schneider's theorem, in turn, is contained in one by D. Ridout, to appear soon in Mathematika. Using the latter, Mahler has settled a basic question concerning Waring's problem.) The above theorem is applied to demonstrate the transcendence of values of a variety of series, perhaps the most interesting being of a type considered by Mahler twenty years ago, which includes the decimal fraction $0.12345678910111213 \cdots$.

Chapter II is primarily concerned with applications of the following theorem: *Let $f_\nu(z)$ $(\nu=1, 2)$ be meromorphic functions with the following properties:*

(a) *Each $f_\nu(z)$ can be represented as a quotient of entire functions of orders at most μ;*

(b) *Each $f_\nu(z)$ satisfies a differential equation of the form $f^{(k)}(z) = P_\nu(f(z), f'(z), \cdots, f^{(k-1)}(z))$, where P_ν is a polynomial with coefficients in an algebraic number field K of degree s;*

(c) *The $f_\nu(z)$, and all their derivatives, assume values in K at the distinct points $z_0, \cdots, z_{m-1}$.*

Then if $m>(2\mu+1)(3s-1/2)$, there must be an algebraic relation between $f_1(z)$ and $f_2(z)$.

This theorem is a descendant (albeit some generations removed, mathematically) of one due to G. Pólya, to the effect that 2^z is in a certain sense the least rapidly increasing integral transcendental function which assumes rational integral values for $z=1, 2, \cdots$. Gel'fond was the first to use such theorems for transcendence problems. Choosing $f_1(z)=z$, $f_2(z)=e^{\alpha z}$ ($\alpha\neq 0$, algebraic) and $z_\lambda=\lambda$ $(\lambda=0, 1, 2, \cdots)$, it follows immediately, since the conclusion of the above theorem is false, that e^α is transcendental. With $f_1(z)=e^z$, $f_2(z)=e^{\beta z}$ (β irrational) and $z_\lambda=\lambda \log \alpha$ ($\alpha\neq 0, 1$) it is equally easy to prove the generalization of Euler's conjecture given by Hilbert as the seventh in his famous list of problems: if α is not 0 or 1, and β is irrational, then at least one of α, β, α^β is transcendental. Many statements about values of elliptic functions, elliptic integrals and modular functions are also deduced, and mention is made of generalizations involving functions of several complex variables and leading to theorems about abelian functions and abelian integrals.

In Chapter III the mutually related classifications of complex numbers due to Mahler and J. F. Koksma are developed. In each scheme, every complex number belongs to one of four classes, according to the exactness with which it can be approximated by algebraic numbers. Elements from different classes are algebraically independent; one class consists of the algebraic numbers themselves; another, containing generalized Liouville numbers, is uncountable but of measure zero; the third may be empty; and the fourth contains almost all real numbers, and almost all complex numbers. The relation between the two classifications is worked out in detail, and the best result known to date (due to the reviewer) concerning a measure-theoretic conjecture advanced by Mahler, is proved.

If ξ is a transcendental number and $P(z)$ is a nonzero polynomial with rational integral coefficients, then $P(\xi)\neq 0$. How close $P(\xi)$ can be to zero depends of course on how complicated $P(z)$ is, that is, on how large are the degree n and the height H—the maximum of the absolute values of the coefficients in $P(z)$. Given ξ, a function $T(n, H)$ such that $|P(\xi)| \geqq T(n, H)>0$ for all nonzero polynomials P is called a measure of transcendence for ξ. The fourth chapter is devoted to the derivation of measures of transcendence for e (Mahler) and α^β (Gel'fond); e.g., it is shown that

$$| P(e) | > H^{-n-cn^2 \log (n+1)/\log \log H}$$

for an appropriate constant c and for all polynomials P with $n\geqq 1$ and $H>H_0(n)$. Moreover, the following theorem, which is an essential improvement of one due to P. Franklin, is proved: *Let α and β be two complex numbers, with $\alpha\neq 0$, 1 and β irrational. Let η_1, η_2, η_3 be zeros of polynomials of degrees $\leqq n$ and heights $\leqq H$. Then if $k>5$, the inequality*

$$\max (| \alpha - \eta_1| , | \beta - \eta_2| , | \alpha^\beta - \eta_3|) < e^{-\log^k H}$$

is not solvable with arbitrarily large H. This theorem has apparently not been published before.