SOLUTION OF THE DIRICHLET PROBLEM FOR EQUATIONS NOT NECESSARILY STRONGLY ELLIPTIC

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Let \( \mu = (\mu_1, \mu_2, \cdots, \mu_n) \) be a sequence of indices and set

\[
|\mu| = \sum \mu_i, \quad D_\mu = \partial^{|\mu|}/(i\partial x_1)^{\mu_1}(i\partial x_2)^{\mu_2} \cdots (i\partial x_n)^{\mu_n},
\]

where \( \xi = (\xi_1, \xi_2, \cdots, \xi_n) \) is any \( n \)-dimensional vector. The linear partial differential operator

\[
A = \sum_{|\mu|=m} a_\mu(x) D_\mu
\]

with complex coefficients \( a_\mu \) is elliptic at a point \( x \) if

\[
P(x, \xi) = \sum_{|\mu|=m} a_\mu(x)\xi^\mu \neq 0
\]

for all real \( \xi \neq 0 \). It is strongly elliptic there if there is a complex constant \( \gamma \) such that \( \text{Re} \, \gamma P(x, \xi) \neq 0 \) for \( \xi \neq 0 \). Let \( G \) be a bounded domain in \( n \)-space and let \( f \) and \( u_0 \) be smooth complex functions defined in \( G \). The Dirichlet problem \((A, f, u_0)\) is to find a complex function \( u \) such that \( Au = f \) in \( G \) and all derivatives of \( u - u_0 \) of order \( < m/2 \) vanish on the boundary \( \partial G \) of \( G \). Gårding [2] and others have shown that if \( \partial G \) and the coefficients \( a_\mu \) are sufficiently smooth, a unique solution exists provided \( A \) is strongly elliptic and \( a_0 \cdots a \) is large enough.

In this paper we extend the existence theory to include any elliptic operator for \( n > 2 \) and to operators satisfying a root condition [5] if \( n = 2 \). Such operators will be called properly elliptic. For \( m = 2 \) all properly elliptic operators are strongly elliptic, but this is not the case for higher orders. For example, the operator corresponding to

\[
P(x, \xi) = \xi_1^4 + \xi_2^4 - \xi_3^4 + i(\xi_1^2 + \xi_2^2)\xi_3^2
\]

is not strongly elliptic.

**Theorem.** Let \( A \) be properly elliptic and denote its formal adjoint by \( A^* \). Assume that the Dirichlet problem \((A^*, 0, 0)\) has only the solution
$u = 0$. Then for any $f$ and $u_0$ sufficiently smooth the Dirichlet problem $(A, f, u_0)$ has a solution.

**Sketch of Proof.** Without loss of generality, we may assume $u_0 = 0$ and for convenience we assume $f \in C^\infty(G)$. Set

$$(v, w)_s = \sum_{|\alpha| \leq s} \int_G D^\alpha v D^\alpha w dx \quad \|v\|_s^2 = (v, v)_s$$

and let $V$ be the set of all $v \in C^\infty(G)$ having all derivatives of order $< m/2$ vanishing on $\partial G$. Complete $V$ with respect to the norm $\| \cdot \|_m$ and call the resulting Hilbert space $H$. From the assumptions on $A$ and $A^*$ it follows [5] that

$$c^{-1} \|v\|_m \leq \|A^*v\|_0 \leq c \|v\|_m$$

for all $v \in H$.

Hence, by the Lax-Milgram lemma [3] there is a $g \in H$ such that

$$(A^*g, A^*v)_0 = (f, v)_0$$

for all $v \in H$.

Applying the regularity theory of Nirenberg [4] and Browder [1], we see that $g \in C^\infty(G)$. Hence $AA^*g = f$ in $G$. Set $u = A^*g \in C^\infty(G)$. Then $Au = f$ in $G$ and

$$(u, A^*v)_0 = (Au, v)_0$$

for all $v \in H$.

This last equality implies $u \in H$. The proof is thus complete.

The foregoing method can also be applied to systems of equations and to general boundary problems which cover $A$ in the sense of [6].

**References**


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