AN ACTION OF A FINITE GROUP ON AN $n$-CELL WITHOUT STATIONARY POINTS

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If $G$ is a transformation group on a space $X$, then $x \in X$ is a stationary point if $gx = x$ for every $g \in G$. It has been an open problem, proposed by Smith [5] and by Montgomery [1, Problem 39], to determine whether every compact Lie group acting on a cell or on Euclidean space has a stationary point. Smith [4; 5] has shown the answer to be in the affirmative in case $G$ is a toral group or a finite group of prime power order. In this note we give a simplicial action of $A_5$, the group of even permutations on five letters, on an $n$-cell without stationary points. Greever [3] has recently shown that the only finite groups of order less than 60 which could possibly act simplicially on a cell without stationary points are a certain class of groups of order 36.

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1. The coset space $SO(3)/I$. Let $SO(3)$ denote the group of all proper rotations of Euclidean 3-space $E^3$ and let $I \subset SO(3)$ be the group of rotational symmetries of the icosahedron. As a group, $I$ is isomorphic to $A_5$ (see [9, pp. 16-18]) and hence is simple.

**Lemma 1.** The coset space $SO(3)/I$ has the integral homology groups of the 3-sphere $S^3$.

**Proof.** Let $Q$ denote the algebra of quaternions and $Q_1 \subset Q$ the group of quaternions of norm one. Identify $Q$ with $E^4$ and $Q_1$ with $S^3$. Let $\tau: Q_1 \rightarrow SO(3)$ be the standard homomorphism, which is a two-to-one covering map. Set $I' = \tau^{-1}(I)$. Then $\tau$ induces a homeomorphism $Q_1/I' \approx SO(3)/I$.

The natural map $\pi: Q_1 \rightarrow Q_1/I'$ is a covering map and the group of covering translations is given by the action of $I'$ on $Q_1$ by right multiplication. Since every covering translation preserves orientation it follows that $Q_1/I'$ is an orientable 3-manifold and hence $H_3(Q_1/I') \approx H_3(SO(3)/I) \approx Z$ (here $Z$ denotes the integers).

From covering space theory the fundamental group $\pi_1(Q_1/I')$ is isomorphic to $I'$. Thus $H_1(Q_1/I')$ is isomorphic to $I'/[I', I']$ where $[I', I']$ denotes the commutator subgroup of $I'$. Since $I$ is simple,
Let $I' = [I', I']$. Also $\tau$ maps $[I', I']$ onto $[I, I]$; it follows that either $[I', I'] = I'$ or $[I', I'] \approx I$. But $Q_1$ contains only one element of order two. Since $I$ contains fifteen elements of order two, $[I', I']$ is not isomorphic to $I$. Thus $I' = [I', I']$ and $H_1(Q_1/I') = 0$. By Poincare duality it follows that $H_2(Q_1/I') = 0$. The lemma follows.

2. Action of $I$ on $SO(3)/I$. Let $I$ act on $SO(3)/I$ by $g_1 \cdot (gI) = g_2gI$. A point $g = gl$ of $SO(3)/I$ is fixed under this action if and only if $g$ belongs to the normalizer of $I$ in $SO(3)$. But $I$ is a maximal finite subgroup of $SO(3)$ (see [9, pp. 16-18]); furthermore, $I$ is not included in any nonfinite proper closed subgroup of $SO(3)$, since this is not the case for the only two classes of such subgroups. Since $I$ is not normal, it follows that $I$ is its own normalizer. Hence there is exactly one stationary point of this action, and this is $\hat{e}$.

We say that the transformation group $G$ acts simplicially on the space $X$ if there exists a triangulation of $X$ with respect to which the homeomorphism $g: X \to X$ is simplicial for every $g \in G$.

**Lemma 2.** The action of $I$ on $SO(3)/I$ is simplicial.

**Proof.** Let $I' \times I'$ act on $Q(=E^4)$ by the rule $(q_1, q_2) \cdot g = q_1gq_2^{-1}$. This represents $I' \times I'$ as a finite group of orthogonal transformations of $E^4$. Hence we may find a triangulation of $S^3(=Q_1)$ such that the action of $I' \times I'$ is simplicial. The method is similar to one used by Whitney [8, p. 358, Lemma 3b]; we omit the details.

Now $e \times I'$ acts simplicially on $Q_1$, and the orbit space is $Q_1/I'$. By taking a barycentric subdivision, the triangulation of $Q_1$ induces a triangulation of the orbit space $Q_1/I'$. The action of $I' \times e$ on $Q_1$ induces an action of $I' \times e$ on $Q_1/I'$ and since $I' \times e$ acts simplicially on $Q_1$, the induced action is simplicial with respect to the induced triangulation of $Q_1/I'$.

In the action of $I' \times e(=I')$ on $Q_1/I'$ the effective group is $I'/\ker \tau$. Furthermore the homeomorphism $\tau_1$ of $Q_1/I'$ on $SO(3)/I$ is equivariant with respect to the action of $I'/\ker \tau$ on $Q_1/I'$ and the action of $I$ on $SO(3)/I$. It follows that the action of $I$ on $SO(3)$ is simplicial.

3. Action of $I$ on a cell. We may assume that the triangulation of $Q_1$ is $C^1$ in the sense of [6] and that $e$ is a vertex. Since

$$\tau_1 \cdot \pi: Q_1 \to SO(3)/I$$

is a $C^1$-map the induced triangulation of $SO(3)/I$ is a $C^1$ triangulation. It follows that the closed star of the point $I$ of $SO(3)/I$ is a 3-cell (see [6, p. 818, Theorem 5]). Let $K$ denote the complex resulting if
we remove the open star of the point $I$ from $SO(3)/I$, and let $|K|$ denote the corresponding space. Then $|K|$ is acyclic (i.e. $H_i(|K|) = 0$ for $i > 0$, and $H_0(|K|) \approx \mathbb{Z}$), and $I$ acts simplicially on $|K|$ without stationary points.

Consider now the join $L = K \circ I$ of the complex $K$ and the complex $I$, where $I$ is the complex consisting of 60 vertices (the points of $I$) and no simplices of higher dimension. Since $I$ acts on $K$, and $I$ acts on $I$ (by left multiplication), then $I$ acts simplicially on $L$. In fact, $g \in I$ maps a line segment from $x \in K$ to $h \in I$ linearly into the line segment from $g(x)$ to $gh$. Furthermore, there are no stationary points on $L$. The polyhedron $|L|$ is a union of 60 cones over $|K|$, each pair intersecting in $|K|$. It follows that $|L|$ is acyclic, and also simply connected.

Let $(v_1, \cdots, v_n)$ denote the set of vertices of $L$. Each $g \in I$ induces a permutation $\eta_g$ of the vertices of $L$; $\eta_g$ may be considered as an element of the full symmetric group $S_n$ on $n$ letters.

Let $e_1, \cdots, e_n$ be basis vectors for $E^n$. Each element $n$ of $S_n$ determines a permutation of $(e_1, \cdots, e_n)$. If we extend linearly, $n$ defines a linear transformation of $E^n$. This defines an action of $S_n$ as a group of linear transformations of $E^n$.

Triangulate $E^n$ so that the action of $S_n$ is simplicial, and so that the simplex spanned by $e_1, \cdots, e_n$ is a simplex of the triangulation. Define an embedding $f$ of $L$ in $E^n$ by setting $f(v_i) = e_i$ and extending $f$ linearly to each simplex. Then $f$ is equivariant. Hence $I$ acts on $f(L)$, and without stationary points.

Let $F_I$ be the set of points of $E^n$ which are stationary under the action of $I$. Then $F_I \cap f(L) = \emptyset$. If we take sufficiently fine barycentric subdivisions we may assume that $F_I$ does not intersect the first closed regular neighborhood of $f(L)$ (see [2, pp. 70–72 for definitions]), denoted by $N(f(L))$. Since $I$ acts simplicially on $E^n$ and $f(L)$ is invariant, it follows that $N(f(L))$ is also invariant. Since $f(L)$ is simply connected and acyclic, it follows from a theorem of J. H. C. Whitehead [7, Corollary 3, p. 298] that the regular neighborhood is a combinatorial $n$-cell. Thus $I$ acts simplicially on the combinatorial $n$-cell $N(f(L))$ without stationary points.

**Bibliography**


**University of Virginia and**

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