SOME PROPOSITIONS EQUIVALENT TO THE CONTINuum HYPOTHESIS

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Let $\mathbb{R}$ denote the real line. If $T \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set $\{t + r : t \in T\} = T[r]$. In [1] we have proved these two theorems:

(\text{B}_K) Let $S \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, $S$ be at most enumerable and $T$ be of first category. Then $\mathbb{R}$ contains a residual subset $R$ such that $S \cap T[r]$ is empty for every $r \in R$.

(\text{B}_M) Let $S \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, $S$ be at most enumerable and $T$ be of measure zero. Then $\mathbb{R}$ contains a subset $R$ such that $\mathbb{R} - R$ is of measure zero and $S[r] \subseteq T[r]$ is empty for every $r \in R$.

We introduce the following propositions:

(\text{K}_3) Let $S \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, $S$ be of power less than $2^{\aleph_0}$ and $T$ be of first category. Then $\mathbb{R}$ contains a residual subset $R$ such that $S \cap T[r]$ is empty for every $r \in R$.

(M_3) Let $S \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, $S$ be of power less than $2^{\aleph_0}$ and $T$ be of measure zero. Then $\mathbb{R}$ contains a subset $R$ such that $\mathbb{R} - R$ is of measure zero and $S[r] \subseteq T[r]$ is empty for every $r \in R$.

(M_3) Let $S \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, $S$ be of power less than $2^{\aleph_0}$ and $T$ be of measure zero. Then there exists an $r \in \mathbb{R}$ such that $S \cap T[r]$ is empty.

Clearly (\text{K}_3) implies (\text{M}_3), (\text{M}_3) implies (\text{M}_*), (\text{K}_*) implies (\text{M}_*), and (\text{M}_*) implies (\text{M}_*).

The following five propositions are discussed at some length in [2]:

(H) $2^{\aleph_0} = \mathfrak{c}$.

(\$) The union of less than $2^{\aleph_0}$ subsets of $\mathbb{R}$ of first category is of first category.

(\text{M}) The union of less than $2^{\aleph_0}$ subsets of $\mathbb{R}$ of measure zero is of measure zero.

(\text{M}_*) $\mathbb{R}$ is not the union of less than $2^{\aleph_0}$ subsets of $\mathbb{R}$ of first category.

(\text{M}_*) $\mathbb{R}$ is not the union of less than $2^{\aleph_0}$ subsets of $\mathbb{R}$ of measure zero.

Evidently (H) implies (\$), (M), (\$) implies (\text{M}_*), and (M) implies (\text{M}_*).

By examining the proofs of (\text{K}_3) and (\text{M}_3), it is easy to see that the following lemma is true.

\text{Lemma 1.} (\$) implies (\text{K}_3), (M) implies (\text{M}_3), (\text{M}_*) implies (\text{K}_3), and (\text{M}_*) implies (\text{M}_3).

Now let $\varnothing$ denote the plane provided with a Cartesian coordinate
system having a horizontal $x$-axis and a vertical $y$-axis. If $\Phi$ is a family of horizontal lines (in $\mathcal{P}$), we say that $\Phi$ is of first category (measure zero) if the union of the members of $\Phi$ intersects the $y$-axis in a linear set of first category (measure zero). If $r \in \mathbb{R}$, we denote by $\Phi[r]$ the family of horizontal lines obtained from $\Phi$ as follows: if $L$ is a member of $\Phi$ and intersects the $y$-axis at $y_0$, then the horizontal line that intersects the $y$-axis at $y_0 + r$ is made a member of $\Phi[r]$. We call the families $\Phi[r]$ ($r \in \mathbb{R}$) the translations of $\Phi$.

We introduce also the following propositions:

(\(\mathcal{S}_N\)) There exists a subset $A$ of $\mathcal{P}$ and a family $\Phi$ of horizontal lines such that

(i) $\Phi$ is of first category,

(ii) there is a subset $U$ of $\mathbb{R}$ of second category such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects $A$ in at most $\aleph_0$ points,

(iii) every member of some nonenumerable set of vertical lines intersects $\mathcal{P} - A$ in at most $\aleph_0$ points.

(\(\mathcal{S}_M\)) There exists a subset $A$ of $\mathcal{P}$ and a family $\Phi$ of horizontal lines such that

(i) $\Phi$ is of measure zero,

(ii) there is a subset $U$ of $\mathbb{R}$ of positive exterior measure such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects $A$ in at most $\aleph_0$ points,

(iii) every member of some nonenumerable set of vertical lines intersects $\mathcal{P} - A$ in at most $\aleph_0$ points.

(\(\mathcal{S}_R\)) There exists a subset $A$ of $\mathcal{P}$ and a family $\Phi$ of horizontal lines such that

(i) $\Phi$ is of first category,

(ii) every translation of $\Phi$ contains a horizontal line that intersects $A$ in at most $\aleph_0$ points,

(iii) every member of some nonenumerable set of vertical lines intersects $\mathcal{P} - A$ in at most $\aleph_0$ points.

(\(\mathcal{S}_R^*\)) There exists a subset $A$ of $\mathcal{P}$ and a family $\Phi$ of horizontal lines such that

(i) $\Phi$ is of measure zero,

(ii) every translation of $\Phi$ contains a horizontal line that intersects $A$ in at most $\aleph_0$ points,

(iii) every member of some nonenumerable set of vertical lines intersects $\mathcal{P} - A$ in at most $\aleph_0$ points.

(\(\mathcal{S}_K\)) There exists a subset $A$ of $\mathcal{P}$ and a family $\Phi$ of horizontal lines such that

(i) $\Phi$ is of power less than $2^{\aleph_0}$,
there is a subset $U$ of $E$ of second category such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects $A$ in at most $\aleph_0$ points,

(iii) every member of some nonenumerable set of vertical lines intersects $\Phi - A$ in a linear set of first category.

$(\mathcal{Q}_M^*)$ There exists a subset $A$ of $\Phi$ and a family $\Phi$ of horizontal lines such that

(i) $\Phi$ is of power less than $2^{\aleph_0}$,

(ii) there is a subset $U$ of $E$ of positive exterior measure such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects $A$ in at most $\aleph_0$ points,

(iii) every member of some nonenumerable set of vertical lines intersects $\Phi - A$ in a linear set of measure zero.

Obviously $(\mathcal{Q}_K^*)$ implies $(\mathcal{Q}_K)$ and $(\mathcal{Q}_M^*)$ implies $(\mathcal{Q}_M)$. We remark that Propositions (33) and (38) in [2] imply $(\mathcal{Q}_K^*)$ and $(\mathcal{Q}_M^*)$, respectively.

**Lemma 2.** $(H)$ implies $(\mathcal{Q}_K^*)$, $(\mathcal{Q}_M^*)$, $(\mathcal{Q}_K)$, and $(\mathcal{Q}_M)$.

**Proof.** Suppose that $(H)$ is true. Then [3, p. 9, Proposition P1] there exists a subset $A$ of $\Phi$ such that the intersection of every horizontal line with $A$ is an at most enumerable set and the intersection of every vertical line with $\Phi - A$ is an at most enumerable set; if we let $\Phi$ consist of a single horizontal line, the truth of $(\mathcal{Q}_K^*)$, $(\mathcal{Q}_M^*)$, $(\mathcal{Q}_K)$, and $(\mathcal{Q}_M)$ is apparent.

**Theorem 1.** The conjunction of $(\mathcal{Q}_K^*)$ and $(\mathcal{Q}_K)$ is equivalent to $(H)$.

**Proof.** (a) Assume that $(H)$ is true. Then Lemma 1 implies that $(\mathcal{Q}_K^*)$ is true, and the truth of $(\mathcal{Q}_K)$ follows from Lemma 2.

(b) Assume that $(\mathcal{Q}_K^*)$ and $(\mathcal{Q}_K)$ are true. If $(H)$ is false, then, in view of (iii) of $(\mathcal{Q}_K)$, there exist $p$ vertical lines, with $\aleph_0 < p < 2^{\aleph_0}$, whose union intersects $\Phi - A$ in a set whose orthogonal projection, $S$, on the $y$-axis is of power less than $2^{\aleph_0}$. If $T$ is the intersection of the $y$-axis with the union of the members of $\Phi$, then, by (i) of $(\mathcal{Q}_K)$, $T$ is a linear set of first category, and $(\mathcal{Q}_K)$ implies that $S$ contains a residual subset $R$ with the property that $S \cap T[r]$ is empty for every $r \in R$. This means that, for some $u \in U$, every member of $\Phi[u]$ intersects each of the aforementioned $p$ vertical lines in a point of $A$, which contradicts (ii) of $(\mathcal{Q}_K)$. Consequently, $(H)$ is true.

**Theorem 2.** The conjunction of $(\mathcal{Q}_M^*)$ and $(\mathcal{Q}_M)$ is equivalent to $(H)$.

**Proof.** In the proof of Theorem 1, replace "$(\mathcal{Q}_K^*)"$ by "$(\mathcal{Q}_M)$", 
"(\(\Omega_K\))" by "(\(\Omega_M\))", "first category" by "measure zero," and "residual subset \(R\)" by "subset \(R\) such that \(\delta - R\) is of measure zero."

**Theorem 3.** The conjunction of \((\mathcal{B}_K^*)\) and \((\Omega_K^*)\) is equivalent to (H).

**Proof.** (a) Assume that (H) is true. Then Lemma 1 implies that \((\mathcal{B}_K^*)\) is true, and the truth of \((\Omega_K^*)\) follows from Lemma 2.

(b) Assume that \((\mathcal{B}_K^*)\) and \((\Omega_K^*)\) are true. If (H) is false, then, in view of (iii) of \((\Omega_K^*)\), there exist \(p\) vertical lines, with \(\aleph_0 < p < 2^{\aleph_0}\), whose union intersects \(\varnothing - A\) in a set whose orthogonal projection, \(S\), on the \(y\)-axis is of power less than \(2^{\aleph_0}\). If \(T\) is the intersection of the \(y\)-axis with the union of the members of \(\Phi\), then, by (i) of \((\Omega_K^*)\), \(T\) is a linear set of first category, and \((\mathcal{B}_K^*)\) implies the existence of an \(r \in \mathfrak{B}\) such that \(S \cap T[r]\) is empty. This means that every member of some translation of \(\Phi\) intersects each of the aforementioned \(p\) vertical lines in a point of \(A\), which contradicts (ii) of \((\Omega_K^*)\). Consequently, (H) is true.

**Theorem 4.** The conjunction of \((\mathcal{B}_M^*)\) and \((\Omega_M^*)\) is equivalent to (H).

**Proof.** In the proof of Theorem 3, replace "(\(\mathcal{B}_K^*)\)" by "(\(\mathcal{B}_M^*)\)", "(\(\Omega_K^*)\)" by "(\(\Omega_M^*)\)", and "first category" by "measure zero."

**Theorem 5.** The conjunction of \((\mathfrak{B})\) and \((\Omega_K^*)\) is equivalent to (H).

**Proof.** (a) Assume that (H) is true. Then, as we have remarked above, \((\mathfrak{B})\) is true, and the truth of \((\Omega_K^*)\) follows from Lemma 2.

(b) Assume that \((\mathfrak{B})\) and \((\Omega_K^*)\) are true. If (H) is false, then, in view of (iii) of \((\Omega_K^*)\), \((\mathfrak{B})\) implies that there exist \(p\) vertical lines, with \(\aleph_0 < p < 2^{\aleph_0}\), whose union intersects \(\varnothing - A\) in a set whose orthogonal projection, \(T\), on the \(y\)-axis is a linear set of first category. If \(S\) is the intersection of the \(y\)-axis with the union of the members of \(\Phi\), then, by (i) of \((\Omega_K^*)\), \(S\) is of power less than \(2^{\aleph_0}\), and \((\mathcal{B}_R)\), which follows from \((\mathfrak{B})\) according to Lemma 1, implies that \(\delta\) contains a residual subset \(R\) with the property that \(T \cap S[r]\) is empty for every \(r \in \mathcal{R}\). This means that, for some \(u \in U\), every member of \(\Phi[u]\) intersects each of the aforementioned \(p\) vertical lines in a point of \(A\), which contradicts (ii) of \((\Omega_K^*)\). Consequently, (H) is true.

**Theorem 6.** The conjunction of \((\mathfrak{M})\) and \((\Omega_M^*)\) is equivalent to (H).

**Proof.** In the proof of Theorem 5, replace "(\(\mathfrak{B}\))" by "(\(\mathfrak{M}\))", "(\(\Omega_K^*)\)" by "(\(\Omega_M^*)\)", "first category" by "measure zero," "(\(\mathcal{B}_R)\)" by "(\(\mathcal{B}_M)\)", and "residual subset \(R\)" by "subset \(R\) such that \(\delta - R\) is of measure zero."
REFERENCES


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