The hypercircle in mathematical physics. By J. L. Synge. New York, Cambridge University Press, 1957. 12+424 pp. \$13.50.

In many fields of mathematical physics, scientists are faced with the problem of solving partial differential equations subject to various boundary conditions, and it is only in rare cases that an exact mathematical solution is feasible. In view of these circumstances, recourse must be had to approximate solutions of one sort or another, e.g. by replacing the differential equation by difference equations to be solved numerically, etc. This book is concerned with the description of a method of approximate solution applicable to a wide class of boundary value problems.

On first glancing upon the intriguing title of the book, the following question arises naturally: Just what is the hypercircle? It is therefore a reviewer's first order of business to attempt to give an answer to this question. This will be done here much in the same way (using Schwarz's and Bessel's inequalities) as it is done in the reviewer's paper in Collectanea Mathematica, Barcelona, mentioned in the preface of the book (this paper was based on an earlier note with A. Weinstein).

Consider a real linear vector space with a scalar product; i.e., a set of elements (called "vectors" following custom) which can be added in pairs ("vector addition"), can be multiplied by real numbers ("scalar multiplication"), these two operations obeying the customary rules of vector algebra. Besides, there is a scalar product, i.e., a real number (a, b) is associated with each ordered pair of vectors a and b, which satisfies the rules: $(\alpha a, b)=\alpha(a, b)$; $(a_1+a_2, b)=(a_1, b)+(a_2, b)$; $(a, b)=(b, a)$, for any vectors a and b and any real number α. If, as will be supposed in most of what follows, $(a, a)\geqq 0$ for any vector a, the scalar product is said to be positive semidefinite. In this case the usual notation $|a|$ is employed for the (non-negative) length of the vector a, and one has $|a|^2=(a, a)$.

The basic inequalities will now be obtained. Let w and c be vectors. Then

$$|w|^2 = |(w-c)+c|^2 = |w-c|^2 + |c|^2 + 2(c, w-c);$$

and in view of Schwarz's inequality: $-|c|\,|w-c| \leqq (c, w-c) \leqq |c|\,|w-c|$, so that

$$
\begin{aligned}
[|w-c| - |c|]^2 &= |w-c|^2 + |c|^2 - 2|c|\,|w-c| \\
&\leqq |w|^2 \leqq |w-c|^2 + |c|^2 + 2|c|\,|w-c| \qquad (1)\\
&= [|w-c| + |c|]^2.
\end{aligned}
$$

In practice, this "triangle inequality" arises in a slightly different form, as follows. It is desired to find upper and lower bounds for the square of the length $(v, v)=|v|^2$ of an a priori *unknown* vector v (this vector v may be, for instance, the solution, or the gradient of the solution, of a boundary value problem). However, some information is known about v (say the boundary conditions and the partial differential equation satisfied by v, compare the example below), so that it is possible to find *known* vectors y and z such that $(y-v, z-v)=0$. In order to apply the previous inequality (1), this equality may be rewritten thus: $-(y+z, v)+(v, v)=-(y, z)$. Completing this square, by adding $|(y+z)/2|^2=|y/2|^2+(y, z)/2+|z/2|^2$ to both sides, one obtains

$$|v-(y+z)/2|^2 = |(y-z)/2|^2. \tag{2}$$

Thus, using (1) and (2), putting $w=v$, and $c=(y+z)/2$, one obtains upper and lower bounds for the a priori unknown number (v, v) in terms of the known numbers $|(y+z)/2|^2$ and $|(y-z)/2|^2$:

$$\left(\left|\frac{y+z}{2}\right|-\left|\frac{y-z}{2}\right|\right)^2 \leqq |v|^2 \leqq \left(\left|\frac{y+z}{2}\right|+\left|\frac{y-z}{2}\right|\right)^2. \tag{3}$$

Now, let n be a positive integer, and $w_1, \cdots, w_n$ be a set of n orthonormal vectors (i.e., each vector has length one and the scalar product of any two distinct vectors is zero). Recalling that for *any* w

$$\begin{aligned} |w|^2 &= \left|w-\sum_{g=1}^{n}(w, w_g)w_g\right|^2+\left|\sum_{g=1}^{n}(w, w_g)w_g\right|^2 \\ &= \left|w-\sum_{g=1}^{n}(w, w_g)w_g\right|^2+\sum_{g=1}^{n}(w, w_g)^2, \end{aligned}$$

it follows immediately from (1), upon replacing w by $w-\sum_{g=1}^{n}(w, w_g)w_g$ and c by $c-\sum_{g=1}^{n}(c, w_g)w_g$ that

$$\begin{aligned} &\left[\left(|c|^2-\sum_{g=1}^{n}(c, w_g)^2\right)^{1/2}-\left(|w-c|^2-\sum_{g=1}^{n}(w-c, w_g)^2\right)^{1/2}\right]^2 \\ (1; n) \qquad &\leqq |w|^2-\sum_{g=1}^{n}(w, w_g)^2 \\ &\leqq \left[\left(|c|^2-\sum_{g=1}^{n}(c, w_g)^2\right)^{1/2}+\left(|w-c|^2-\sum_{g=1}^{n}(w-c, w_g)^2\right)^{1/2}\right]^2, \end{aligned}$$

which may be thought of as containing (1) as the special case $n=0$.

In practice, this inequality arises in a slightly different form in estimating the length $(v, v) = |v|^2$ of an a priori unknown vector v. Using only the information available about v it is possible to find, in addition to the known vectors y and z such that $(y-v, z-v) = 0$, a set of $p+q$ known orthonormal vectors $y_1, \cdots, y_p$ and $z_1, \cdots, z_q$ (notice the correspondence by alphabetical order of the notation: g, h, i, j; n, o, p, q; and w, x, y, z) such that $(y_i, z-v) = 0$ for $i = 1, \cdots, p$, and $(y-v, z_j) = 0$ for $j = 1, \cdots, q$. These $p+q$ vectors take the place of the n vectors $w_1, \cdots, w_n$ in (1; n). Putting $w = v$; $c = (y+z)/2$; $n = p+q$; $w_i = y_i$, $i = 1, \cdots, p$; $w_{p+j} = z_j$ for $j = 1, \cdots, q$; a slight computation shows that (in view of (2) above and the fact that $(v, y_i) = (z, y_i)$ for $i = 1, \cdots, p$ and $(v, z_j) = (y, z_j)$ for $j = 1, \cdots, q$):

$$\left|v - \frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(v - \frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(v - \frac{y+z}{2}, z_j\right)^2$$

$$= \left|\left(v - \frac{y+z}{2}\right) + \left\{\sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)y_i - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)z_j\right\}\right|^2$$

(2; p, q)

$$= \left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2.$$

The final result of making these substitutions in (1; n) is:

$$\left[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y+z}{2}, z_j\right)^2\right)^{1/2} - \left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{1/2}\right]^2$$

(3; p, q)

$$\leq |v|^2 - \sum_{i=1}^{p}(z, y_i)^2 - \sum_{j=1}^{q}(y, z_j)^2$$

$$\leq \left[\left(\left|\frac{y+z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y+z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{1/2} + \left(\left|\frac{y-z}{2}\right|^2 - \sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2 - \sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right)^{1/2}\right]^2,$$

which may be thought of as containing (3) as the special case $p = q = 0$. Inequality (3; p, q) furnishes numerically computable upper and lower bounds for the unknown number (v, v) in terms of the known vectors y, z, y_i, and z_j.

Inequality (3; p, q), which has been derived here in a few lines, is the "key to the method of the hypercircle" (see pages 118–120 of the book), and its application in a special case, (which reveals the essential steps to be followed in any other case), will be explained next. But, the reader may very well ask himself at this juncture: Just where is the hypercircle? For it may be that a reader of the present derivation of inequality (3; p, q) has missed seeing the trees of the geometry for the woods of the analysis, much as a reader of the book may miss seeing the trees of the analysis for the woods of the geometry. In view of this, a search for the lost hypercircle will now be conducted. Consider equation (2) above. In geometric language, it states that the unknown vector v lies on the "hypersphere" S with center $(y+z)/2$ and radius $|(y-z)/2|$, which consists of all vectors u such that

$$\left|u-\frac{y+z}{2}\right|^2=\left|\frac{y-z}{2}\right|^2.$$

Further, in terms of the $p+q$ unit vectors y_i and z_j occurring in (2; p, q), the unknown vector v lies on the intersection SP (=hypercircle) of the sphere S and the "hyperplane P of class $p+q$" consisting of all vectors u such that $(u, y_i)=(z, y_i)$ for $i=1, \cdots, p$ and $(u, z_j)=(y, z_j)$ for $j=1, \cdots, q$. That is, (compare equation (2; p, q)) the vector v lies on the hypercircle of center

$$\frac{y+z}{2}-\sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)y_i+\sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)z_j$$

and radius

$$\left[\left|\frac{y-z}{2}\right|^2-\sum_{i=1}^{p}\left(\frac{y-z}{2}, y_i\right)^2-\sum_{j=1}^{q}\left(\frac{y-z}{2}, z_j\right)^2\right]^{1/2};$$

there is the hypercircle.

An example will now be given, illustrating the application of the previous inequalities to the estimation of the Dirichlet integral $\int_D(u_\xi^2+u_\eta^2)d\xi d\eta$ of the solution $u(\xi, \eta)$ of a plane Dirichlet problem. The real valued function $u(\xi, \eta)$ satisfies the second order partial differential equation $u_{\xi\xi}+u_{\eta\eta}=F(\xi, \eta)$ in a bounded plane domain D with a smooth boundary C; and satisfies the boundary condition $u=f$, on C, where F and f are given real valued functions on D and on C, respectively. It only remains to specify precisely what the scalar product is, and what the "vectors" v, y, z, y_i, z_j stand for in this special case, and the previous inequalities will then be self explanatory. The linear vector space may be chosen to consist here of all ordered pairs $[p_1, p_2]$ of sufficiently smooth real valued functions defined on $D+C$. Vector addition and multiplication of a vector by a real number are defined by the equations: $[p_1, p_2]+[q_1, q_2]=[p_1+p_2, q_1+q_2]$ and $\alpha[p_1, p_2]=[\alpha p_1, \alpha p_2]$; while the scalar product is defined by $([p_1, p_2], [q_1, q_2])\equiv\int_D(p_1q_1+p_2q_2)d\xi d\eta$. The unknown vector v is just $[\partial u/\partial\xi, \partial u/\partial\eta]$, and upper and lower bounds are sought for the a priori unknown number

$$|v|^2=\left(\left[\frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right], \left[\frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right]\right)=\int_D\left\{\left(\frac{\partial u}{\partial\xi}\right)^2+\left(\frac{\partial u}{\partial\eta}\right)^2\right\}d\xi d\eta,$$

the Dirichlet integral of u. The essential fact used in the selection of the various vectors to be chosen is the following orthogonality relation: *if* $[p_1, p_2]=[\partial u'/\partial\xi, \partial u'/\partial\eta]$, that is, $[p_1, p_2]$ is the piecewise continuous gradient of a continuous function $u'(\xi, \eta)$ which *vanishes* on C; and if the *normal* component $q_1n_1+q_2n_2$ is continuous across any curve in D, while q_1 and q_2 are piecewise continuous and satisfy $\partial q_1/\partial\xi+\partial q_2/\partial\eta=0$ in D; *then* $([p_1, p_2], [q_1, q_2])=\int_D(p_1q_1+p_2q_2)d\xi d\eta=0$, by Green's theorem. The vector $z=[\partial z/\partial\xi, \partial z/\partial\eta]$ is the gradient of a sufficiently smooth function $z(\xi, \eta)$ such that $z=u=f$ on the boundary C, while $z_j=[\partial z_j/\partial\xi, \partial z_j/\partial\eta]$, with $z_j=0$ on C, for $j=1, \cdots, q$. Further, $y=[y_1, y_2]$, with $\partial y_1/\partial\xi+\partial y_2/\partial\eta=F$, in D, while $y_i=[y_{i1}, y_{i2}]$ with $\partial y_{i1}/\partial\xi+\partial y_{i2}/\partial\eta=0$, in D, for $i=1, \cdots, p$. Inequality (3; p, q) is then entirely self explanatory.

So far, mention has only been made of the estimation of a quadratic integral, (like the Dirichlet integral in the example above), of the solution of a boundary value problem, which is taken care of by Schwarz's and Bessel's inequalities, as has been shown. But it is also of interest to estimate the value of the unknown solution of a boundary value problem (or of the derivatives of such a solution) at a given point. This can also be readily taken care of by Schwarz's and Bessel's inequalities, once it is shown that the desired unknown number is equal to a suitable scalar product (w, d), where d is a known vector and w is a priori unknown. If c is any vector (known, in practice) then one has from Schwarz's and Bessel's inequalities that $(w-c, d)^2\leqq(w-c, w-c)(d, d)$ and $(w-c, d)^2\leqq(w-c, w-c)\left[(d, d)-\sum_{g=1}^{n}(d, w_g)^2\right]$, where the number $(w-c, w-c)$ (compare equation (2) above) and the vectors $w_1, \cdots, w_n$ are known.

After this detailed discussion of the basic elementary ideas, only a brief account of the general outline of the book is needed to *round* out the picture of the contents of the book. There are three main parts, entitled: No metric, Positive-definite metric and Indefinite metric. In the terminology of the present review the titles of the three parts correspond respectively to: (real) Linear vector spaces, Linear vector spaces with a positive definite scalar product, and Linear vector spaces with a scalar product, not necessarily positive definite. Part I (= Chapter 1) is mainly of an introductory nature. Part II is concerned with geometrical considerations in a linear vector space with a positive definite scalar product. Chapter 2 ends with a section entitled "The key to the hypercircle method," and this is where the present review started. Chapter 3 bears the title: "The Dirichlet problem for a finite domain in the Euclidean plane." Chapter 4 is titled "The torsion problem." Chapter 5 deals with various boundary value problems, for example, the equilibrium of an elastic body. Part III contains two chapters, one on geometry and the other one on vibration problems. Somewhat loosely phrased, the general idea is that the minimum principles of Part II become variational principles in Part III.

The printing and format of the book are excellent. The exposition is of the highest order; many an exquisitely turned phrase is to be found among its pages. There is a wealth of figures and every section ends with a set of exercises for the reader. The author's keen concern for actual numerical results is evident from the many specific examples which he has worked out in detail, using a hand computer. Of special interest in this regard is his clear distinction, at the end of Chapter 2, between reliable and unreliable bounds, relative to the usual methods of numerical computation to so many significant figures. The author's point about "the practical computer (who claims to have *solved* a set of equations, when he has not, strictly speaking)" is very well taken.

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Differentialgeometrie. By E. Kreyszig. Mathematik und ihre Anwendungen in Physik and Technik. Series A, vol. 25. Leipzig, Akademische Verlagsgesellschaft Geest und Portig K. G., 1957, 9+421 pp. DM 36.

This book belongs to the type, represented in English by Eisenhart's *Introduction to differential geometry* (Princeton, 1940), in French by Bouligand's *Principes de l'analyse géométrique* (Paris, 3d ed., 1949) and in German and Čech by Hlavaty's *Differentialgeometrie* (Groningen, 1939), written for those who believe that the standard material of classical differential geometry is best presented within the context of or at any rate together with the tensor calculus. Such students and lecturers will find in the present book a pleasant and unhurried presentation of the elementary theory of curves and surfaces