HOMOMORPHISMS AND IDEMPOTENTS OF GROUP ALGEBRAS

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Let $G$ be a locally compact abelian group. We denote by $M(G)$ the algebra of all finite complex-valued Borel measures on $G$. The algebra is normed by assigning to each measure its total variation, and the product or convolution of the measures $\mu$ and $\nu$ is defined by

$$(\mu * \nu)(E) = \int \int_{x+y \in E} d\mu(x)d\nu(y).$$

If a particular Haar measure is chosen on $G$, the subalgebra of absolutely continuous measures may be identified with $L(G)$, the algebra of absolutely integrable functions. The Fourier transform of a measure $\mu$ is a function $\hat{\mu}$ defined on $\hat{G}$, the dual group of $G$, by the formula

$$\hat{\mu}(x) = \int_{\hat{G}} (\chi, g) d\mu(g),$$

where $(\chi, g)$ denotes $\chi$ evaluated at $g$. Each $\chi$ thus yields a homomorphism of $M(G)$ onto the complex numbers. Every such homomorphism of $L(G)$ is obtained in this way.

Let $\Phi$ be a homomorphism of $L(G)$ into $M(H)$. After composing with $\Phi$, every homomorphism of $M(H)$ onto the complex numbers either is identically zero, or can be identified with a member of $\hat{G}$. We thus have a map $\phi$ from $\hat{H}$ into $\{\hat{G}, 0\}$, the union of $\hat{G}$ and the symbol 0, the latter to be considered as the point at infinity. Our main result is:

**Theorem 1.** For every homomorphism $\Phi$ of $L(G)$ into $M(H)$, there exist a finite number of cosets of open subgroups of $\hat{H}$, which we denote by $K_i$, and continuous maps $\psi_i: K_i \rightarrow \hat{G}$, such that

$$\psi_i(x + y - z) = \psi_i(x) + \psi_i(y) - \psi_i(z),$$

with the following property: there is a decomposition of $\hat{H}$ into the disjoint union of sets $S_j$, each lying in the Boolean ring generated by the sets $K_i$, such that on each $S_j$, $\phi_*$ is either identically zero or agrees with some $\psi_i$, where $S_j \subseteq K_i$.

Conversely, for any such map of $\hat{H}$ into $\{\hat{G}, 0\}$, there is a homo-
morphism of \( L(G) \) into \( M(H) \) which induces it. The map carries \( L(G) \) into \( L(H) \) if and only if \( \phi^{-1}_* \) of every compact subset of \( \hat{G} \) is compact.

The main tool in the proof of Theorem 1 is the following lemma:

**Lemma.** If \( G \) and \( H \) are compact, then the graph of \( \phi_* \), namely all pairs \((\phi_*(h), h)\) where \( \phi_*(h) \) is not zero, is such that its characteristic function is the Fourier transform of a measure on \( G \times H \).

The measure in the lemma must of course be an idempotent, that is, satisfy the equation \( \mu \ast \mu = \mu \). The essential difficulty rests in the determination of all idempotent measures on a group.

**Theorem 2.** If \( \mu \) is an idempotent measure, then \( \mu \) is the characteristic function of a subset \( E \) of \( \hat{G} \) which lies in the Boolean ring generated by cosets of open subgroups of \( \hat{G} \).

It is not difficult to deduce Theorem 1 from the above statements in the case in which \( G \) and \( H \) are compact. In the general case one shows that there is a natural extension of \( \phi \) to a homomorphism of \( L(\hat{G}) \) into \( M(\hat{H}) \) where \( \hat{G} \) and \( \hat{H} \) are the Bohr compactifications of \( G \) and \( H \) respectively. It can then be shown that if \( \hat{G} \) and \( \hat{H} \) are taken in the discrete topology, Theorem 1 holds. However we know that \( \phi_* \) is continuous and after some manipulation we can show that Theorem 1 holds in the original form.

Both Theorems 1 and 2 were known in special cases before. We note that Theorem 2 implies that the support of an idempotent measure is contained in a compact subgroup. Conversely, it is simple to reduce Theorem 2 to the case where \( \hat{G} \) is compact. If \( \mu \) is absolutely continuous then it clearly is a finite sum of characters multiplied by Haar measure. The difficulty in general lies in analyzing the singular part of \( \mu \). Here the main point is to show that \( \mu \) has mass on a closed subgroup of infinite index. In the case that \( \hat{G} \) has no elements of finite order, this statement is equivalent to saying that the set \( E \) intersects some cyclic subgroup of \( \hat{G} \) in an infinite set. For arbitrary \( \hat{G} \) it is proved by more complicated means. In either case one needs a technique which will yield some restriction on the nature of the set \( E \). It is of course true that \( E \) can be an arbitrary finite set. Hence we can only hope to derive statements about the set \( E \) which allow for a finite number of exceptions. Nevertheless, our technique yields statements concerning finite sums of characters. These we state for the circle group.

**Theorem 3.** For some \( K \), whenever \( c_j \) are such that \( |c_j| \geq 1 \), and \( n_j \) are arbitrary distinct integers, we have
\[ \int_0^{2\pi} \left| \sum_{j=1}^N c_j e^{in_j x} \right| dx > K \left( \frac{\log N}{\log \log N} \right)^{1/8}. \]

It is a conjecture of Littlewood that the inequality holds with \( K \log N \) on the right side. Previously, however, it was not even shown that the left side tended to infinity as a function of \( N \). Indeed in the course of the proof of Theorem 2 we actually need this fact. The proof of Theorem 3 is completely independent of any abstract considerations. It is accomplished by exhibiting finite linear combinations of exponentials, \( \phi_k \), such that \( |\phi_k| \leq 1 \) and yet, if \( \mu \) denotes the measure

\[ \sum c_j e^{in_j x} dx, \]

\( \int \phi_k d\mu \) is large. We use some general lemmas concerning measures together with a combinatorial argument concerning the distribution of the integers \( n_j \). In the case of idempotent measures, the same type of argument is used to show that the set \( E \) has many finite sets \( P \) such that for all \( x \) in \( E \), there is some \( p \) in \( P \) such that \( x + p \) lies in \( E \). This, however, does not suffice to characterize \( E \) and further arguments are necessary. The details are too complicated to give here but will appear in forthcoming publications.

References