EXTENSIONS OF THE LEMMA OF HAAR IN THE CALCULUS OF VARIATIONS

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This note is concerned with necessary and sufficient conditions on the coefficients \( A_i \) in order that a linear functional of the form

\[
L(v) = \int_G \sum_i A_i D^i v dx
\]

shall vanish identically on a suitable class of functions \( v \) which vanish on the boundary \( G^* \) of the connected open set \( G \) in \( n \)-dimensional \( x \)-space. Here \( i \) denotes an \( n \)-dimensional vector with nonnegative integer components \( i_j \), and

\[
D^i v = \prod_{j=1}^{n} D_{x_j}^i v_j
\]

where \( D_{x_j} \) denotes partial differentiation with respect to \( x_j \). The sum in (1) is taken over all vectors \( i \) with \( 0 \leq i_j \leq m_j \), where \( m \) is a fixed vector with positive integer components.

For the domain of the functional \( L \) it is convenient to take the class of all functions \( v \) of class \( C^\infty \) and having support compact on \( G \) (i.e., compact and contained in \( G \)). Then \( L(v) \) is well defined when the coefficients \( A_i \) are all locally integrable in \( G \). Also the following notations are meaningful (with exceptional sets of measure zero) for a locally integrable function \( f \):

\[
M_{x_jh_j} f(x) = \int_0^h f(y) ds, \quad \Delta_{x_jh_j} f(x) = f(x) - f(x),
\]

where \( y_j = x_j + s, \ z_j = x_j + h_j, \ y_k = z_k = x_k \) for \( k \neq j \), and

\[
M_h^i = \prod_{j=1}^{n} M_{x_jh_j}^i, \quad \Delta_h^i = \prod_{j=1}^{n} \Delta_{x_jh_j}^i.
\]

We understand that \( x \) is a point in \( G \), and that \( h \) is taken so small that all the points \( x + ih = (x_j + i_jh_j) \) considered lie in \( G \). We also set

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The form \( L(v) \) with coefficients \( A_i \) locally integrable in \( G \) vanishes for all function \( v \) in the class \( C^\infty \) and having support compact in \( G \) if and only if \( H_hA = 0 \) for all intervals \( [x, x+h] \) contained in \( G \) except those for which one of the points \( x+ih \) with \( 0 \leq i_j \leq m_j \) lies in a set \( E \) of measure 0.

The proof of the necessity of the condition begins by observing that when the coefficients \( A_i \) are sufficiently smooth, a suitable integration by parts shows that the Euler expression

\[
EA = \sum_i (-1)^{|i|} D^i A_i
\]

vanishes on \( G \). Then by use of integral means \( M^n h^j A \) and the formula \( EM^n h^j A = H_hA \), we proceed to the case when the \( A_i \) are merely continuous. Finally, by another application of integral means we arrive at the general case.

Another condition for the vanishing of \( L(v) \) when derivatives of order higher than the first appear was given by Hilbert in 1904 (Math. Ann. vol. 59, pp. 166–168) for the case when \( n = 2 \) and \( m_1 = m_2 = 3 \), and the coefficients \( A_i \) are continuous. Extensions to other cases were proved by Mason and others. The Hilbert-Mason form of the condition may be derived from the extended form of the Haar lemma as indicated below. Assuming that the shape of the region \( G \) is suitably restricted and that \( a \) is a point of \( G \), we set

\[
I_{x,j}(x) = \int_{a_j}^{x_j} f(t) dt, \quad \text{where} \quad t_k = x_k \text{ for } k \neq j,
\]

\[
I^i = \prod_{j=1}^{n} I_{x,j}^{i_j}
\]

\[
RA = \sum_i (-1)^{|i|} I^{m-i} A_i.
\]

It is readily seen that

\[
\Delta_h^m RA = H_hA,
\]

and then with the help of integral means it may be shown that the
condition $H_h A = 0$ almost everywhere (as stated in the theorem above) is equivalent to the following condition:

RA is equal almost everywhere in $G$ to a sum of $n$ functions $c_h(x)$, where $c_h(x)$ is a polynomial of degree less than $m_h$ in $x_h$, with coefficients which are locally integrable functions of the remaining variables.

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