CIRCUMSCRIBED CUBES IN EUCLIDEAN $n$-SPACE

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Let $E^n$ be a euclidean $n$-space with a rectangular cartesian coordinate system $(x) = (x_1, \ldots, x_n)$, and let $(y)$ be any system which is a rotation of $(x)$. Let $A \subseteq E^n$ be a closed bounded set containing $n+1$ linearly independent points. Its circumscribed $(y)$-box is the set $[a_i, b_i] \subseteq \mathbb{R}$ ($i = 1, \ldots, n$) where $a_i$ and $b_i$ are the respective minimum and maximum values of $y_i$ on $A$. Let $c_i = b_i - a_i$ be interpreted as a function on the space $\mathbb{R}^{n-1}$ of rotations of coordinate systems, which is also the rotation space of the unit $(n-1)$-sphere $S^{n-1} \subseteq E^n$.

Let $f: \mathbb{R}^{n-1} \to E^n$ be the function which maps $r \in \mathbb{R}^{n-1}$ onto the point $(c_1(r), \ldots, c_n(r))$, relative to the fixed initial coordinate system $(x)$. Let $D$ be the diagonal $x_1 = \ldots = x_n$ in $E^n$. The circumscribed $(y)$-box corresponding to a point $r \in \mathbb{R}^{n-1}$ is an $n$-cube if and only if $f(r) \in D$. Accordingly, $K = f^{-1}(D)$, a subspace of $\mathbb{R}^{n-1}$, will be called the space of circumscribed $n$-cubes of $A$. Its structure can be studied by means of the mapping $f$. For the purpose of this study the significant properties are as follows: (1) $f$ is a continuous mapping of $\mathbb{R}^{n-1}$ into the region $x_i > 0$ ($i = 1, \ldots, n$) of $E^n$ (2) $f(\mathbb{R}^{n-1})$ is symmetric with respect to $D$. This second property follows from the fact that all possible permutations of axial directions can be achieved in a symmetric way through rotations. There is no need to distinguish between the two possible senses on a given $y_i$-direction, since the value of $c_i$ is the same for both. Hence, one gets odd as well as even permutations of the $c_i$'s.

Let $T^{n-1}$ be the simplex in $E^n$ with vertices at the unit points on the $(x)$-axes. A central projection from the origin carries the mapping $f$ into a continuous mapping $g: \mathbb{R}^{n-1} \to T^{n-1}$ where $g(\mathbb{R}^{n-1})$ is symmetric in the barycentric coordinates on $T^{n-1}$. The inverse image $g^{-1}(q)$, where $q$ is the barycenter of $T^{n-1}$, is identical with $f^{-1}(D) = K$. This leads to the following result.

**Theorem.** The space of circumscribed cubes of a closed subset of euclidean $n$-space containing $n+1$ independent points is the inverse image $K = g^{-1}(q)$ of the center of an $(n-1)$-simplex $T^{n-1}$ under a continuous mapping $g: \mathbb{R}^{n-1} \to T^{n-1}$, where $\mathbb{R}^{n-1}$ is the rotation space of an $(n-1)$-sphere and where $g(\mathbb{R}^{n-1})$ is symmetric in the barycentric coordinates on $T^{n-1}$.

Any particular circumscribed $n$-cube is the $(y)$-cube for a system $(y)$ obtainable from $(x)$ without rotating any axis by more than
\[\pi/2.\] A smaller number than \(\pi/2\) can be used. Thus all the \(n\)-cubes, but not all the rotations yielding them, correspond to points of \(K \cap Q^m\) where \(Q^m \subset R_n\) is an \(m\)-cell of dimension \(m = (n^2 - n)/2\), the dimension of \(R_n\), and where \(Q^m\) can be so selected that \(g(Q^m)\) is symmetric in the barycentric coordinates on \(T_n\).

The intersection \(K \cap Q^m\) is the inverse image of \(q^m\) under the symmetric mapping \(g: Q^m \to T_n\) of a closed \(m\)-cell into a closed \((n - 1)\)-cell. This leads to the conclusion that \(K\) is of dimension at least \(m - (n - 1) = (n - 1)(n - 2)/2\), which is the dimension of the rotation space \(R_{n-2}\) of the \((n - 2)\)-sphere. It is conjectured that \(K\) contains a subspace homeomorphic to \(R_{n-2}\), a conjecture easily established for \(n = 3\). Thus, there exists at least a 1-circuit of circumscribed cubes in the three-dimensional case.

These results strengthen the existence theorems of Kakutani [1] in three dimensions and of Yamabe and Yujobô [2] in \(n\) dimensions.

The above is an outline of material being written up in detail for publication elsewhere.

Bibliography


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