AN ARITHMETICAL INVERSION PRINCIPLE

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Let \( f(n, r) \) represent an even function of \( n \) (mod \( r \)); that is, \( f(n, r) = f((n, r), r) \) for all integers \( n \) and a positive integral variable \( r \). The following inversion relation is proved in [2]. If \( r = r_1r_2 \) and \( f(n, r) \) is even (mod \( r \)), then

\[
g(r_1, r_2) = \sum_{d|r_1} f\left(\frac{r_1}{d}, r\right) \mu(d) \Leftrightarrow f(n, r) = \sum_{d|r} g\left(d, \frac{r}{d}\right),
\]

where \( \mu(r) \) denotes the Möbius function. This relation can be easily verified on the basis of the definition of even function (mod \( r \)) and the characteristic property of \( \mu(r) \),

\[
\sum_{d|r} \mu(d) = \varepsilon(r) = \begin{cases} 1 & (r = 1), \\ 0 & (r > 1). \end{cases}
\]

We now state a generalization of (1). Let \( \xi(r) \) and \( \eta(r) \) be arithmetical functions satisfying

\[
\sum_{d|r} \xi(d)\eta(\delta) = \varepsilon(r).
\]

The following theorem can be proved in the same manner as (1), with (3) used in place of (2).

**Theorem 1.** If \( r = r_1r_2 \) and \( f(n, r) \) is even (mod \( r \)), then

\[
g(r_1, r_2) = \sum_{d|r_1} f\left(\frac{r_1}{d}, r\right) \eta(d) \Leftrightarrow f(n, r) = \sum_{d|r} g\left(d, \frac{r}{d}\right) \xi(\delta).
\]

Clearly (4) reduces to (1) in case \( \xi(r) = 1, \eta(r) = \mu(r) \). The case \( \xi(r) = \mu(r), \eta(r) = 1 \) yields the following dual of (1).

**Theorem 2.** If \( r = r_1r_2 \) and \( f(n, r) \) is even (mod \( r \)), then

\[
g(r_1, r_2) = \sum_{d|r_1} f\left(\frac{r_1}{d}, r\right) \mu(d) \Leftrightarrow f(n, r) = \sum_{d|r} g\left(d, \frac{r}{d}\right) \xi(\delta).
\]

An immediate consequence of Theorem 2 is

**Corollary 2.1.** For every arithmetical function \( g(r_1, r_2) \) of two positive integral variables \( r_1, r_2 \), there exists a uniquely determined even function (mod \( r \), \( f(n, r) \), such that \( g(r_1, r_2) \) is expressible as a divisor sum (5) with respect to \( f(n, r) \).
The relation (1) is applied in [2] to give a new proof of the Anderson-Apostol generalization [1] of the Hölder formula,

\[ \frac{\phi(r) \mu(m)}{\phi(m)} = \sum_{d \mid (n, r)} d \mu \left( \frac{r}{d} \right), \quad \left( m = \frac{r}{(n, r)} \right), \]

where \( \phi(r) \) represents the Euler \( \phi \)-function. The following analogue of the generalized Hölder relation can be proved in a similar manner, with (5) replacing (1) in the proof.

Let \( g(r) \) and \( h(r) \) denote arithmetical functions, and define

\[ f(n, r) = \sum_{d \mid (n, r)} h(d) g \left( \frac{r}{d} \right) \mu^2 \left( \frac{r}{d} \right) \mu(\delta), \quad F(r) = f(0, r). \]

**Theorem 3.** If \( g(r) \) is multiplicative and \( h(r) \) is completely multiplicative, and if \( f \), for all primes \( p \), \( h(p) \neq 0 \), \( g(p) \neq h(p) \), then

\[ \frac{F(r)g(m)\mu^2(m)}{F(m)} = f(n, r), \quad \left( m = \frac{r}{(n, r)} \right). \]

Application of (8), with \( h(r) = r, g(r) = 1 \), in connection with the Dedekind-Liouville formula,

\[ \phi(r) = \sum_{d \mid r} \mu(d) / d, \]

yields the following analogue of Hölder’s formula (6):

**Corollary 3.1.**

\[ \frac{\phi(r) \mu^2(m)}{\phi(m)} = \sum_{d \mid r, d \mid (n, r)} d \mu^2(e) \mu(\delta), \quad \left( m = \frac{r}{(n, r)} \right). \]

Similarly, with \( h(r) = 1, g(r) = \mu(r) / \phi(r) \) in (8) it follows, on applying Landau’s identity,

\( r / \phi(r) = \sum_{d \mid r} \mu^2(d) / \phi(d) \), that

**Corollary 3.2.**

\[ \frac{(n, r) \mu(m)}{\phi(r)} = \sum_{d \mid (n, r), \phi(d) \mu(\delta)} \frac{\mu(e) \mu(\delta)}{\phi(e)}, \quad \left( m = \frac{r}{(n, r)} \right). \]

Other potentially useful relations can be derived in a similar manner.

**Bibliography**


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