ON INDEPENDENT GROUP CHARACTERS

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The theorem proved in this note, when taken in conjunction with the theory of the Bohr compactification of a locally compact abelian group (for which see [1]), provides density theorems for group characters which generalize the classical Kronecker and Kronecker-Weyl approximation theorems. The theorems thus obtained are in several respects extensions of those of Bundgaard [2]. An account of them will appear elsewhere.

If $G$ is a locally compact abelian group then a character of $G$ will be taken here to mean a continuous homomorphism $\chi$ of $G$ into the circle group $T$. If $G$ is discrete then its character group $H = G^*$ is compact and carries a unique Haar measure $\mu$ such that $\mu(H) = 1$. If $\mathcal{B}$ is the class of Borel subsets of $H$ then $(H, \mathcal{B}, \mu)$ is a probability field in the sense of Kolmogorov [3], and, for each $g \in G$, the function $\chi \mapsto \chi(g)$ on $H$ into $T$ is a character of $H$, and is a fortiori a random variable for $(H, \mathcal{B}, \mu)$.

If $\mathcal{S} \not= S \subseteq G$ then $[S]$ will denote the subgroup of $G$ generated by $S$, except that, if $S = \langle g \rangle$, $[S]$ will also be denoted by $\langle g \rangle$. The symbols $\mathcal{P}$, $\prod$ are used respectively for the restricted and unrestricted direct products. Thus if $(G_\lambda)_{\lambda \in \Lambda}$ is a family of discrete abelian groups then $\mathcal{P}_{\lambda \in \Lambda} G_\lambda$ is discrete, $\prod_{\lambda \in \Lambda} G_\lambda^*$ is compact, and each is the character group of the other for their natural pairing (see [4, §37]).

**Theorem.** Let $S = \langle g_\lambda \rangle_{\lambda \in \Lambda}$ be a nonempty family of elements of $G$, let $K_\lambda = \{ \chi(g_\lambda) \mid \chi \in H \}$ and let $\phi_S : H \to \prod_{\lambda \in \Lambda} K_\lambda$ be the homomorphism $\chi \mapsto (\chi(g_\lambda))_{\lambda \in \Lambda} = \phi_S(\chi)$.

Then the following statements are equivalent:

(i) $[S] = \mathcal{P}_{\lambda \in \Lambda} \langle g_\lambda \rangle$;

(ii) $\phi_S(H) = \prod_{\lambda \in \Lambda} K_\lambda$;

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(iii) the functions $\chi \rightarrow \chi(g_\lambda), \lambda \in \Lambda$, constitute an independent family of random variables for the probability field $(H, \mathcal{B}, \mu)$.

We prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

If (i) is true then $H/[S]^\perp = [S]^* = \prod_{\lambda \in \Lambda} [g_\lambda]^*$, where $[S]^\perp = \{\chi \in H | \chi(g) = 1 \text{ for all } g \in [S]\}$. For each $\chi \in H$ we can therefore find a unique family $(\chi_\lambda)_{\lambda \in \Lambda}$ with $\chi_\lambda \in [g_\lambda]^*$, $\lambda \in \Lambda$, such that $\chi(s) = \prod_{\lambda \in \Lambda} \chi_\lambda(s_\lambda)$ for all $s = \prod_{\lambda \in \Lambda} s_\lambda \in [S]$, where $s_\lambda \in [g_\lambda]$ for $\lambda \in \Lambda$. Condition (ii) follows at once.

Suppose next that (ii) is true. The group $K = \prod_{\lambda \in \Lambda} K_\lambda$ is compact and therefore carries a Haar measure $\nu$ for which $\nu(K) = 1$. The map $\phi_S : H \rightarrow K$ is an epimorphism and therefore $\mu(\phi_S^{-1}(A)) = \nu(A)$ for each Borel set $A \subseteq K$. Now let $\Lambda_0 = (\lambda_1, \lambda_2, \cdots, \lambda_n) \subseteq \Lambda$, where $1 \leq n < \infty$, and let $A_r$ be a Borel subset of $K_{\lambda_r}, 1 \leq r \leq n$, and for each $\lambda \in \Lambda$ let $\nu_\lambda$ be the Haar measure on $K_\lambda$, normalized so that $\nu_\lambda(K_\lambda) = 1$. Suppose also that $B_\lambda = A_r$ for $\lambda = \lambda_r, 1 \leq r \leq n$, and that $B_\lambda = K_\lambda$ for $\lambda \notin \Lambda_0$. Then, if $E_r = \{\chi \in H | \chi(g_\lambda) \subseteq A_r\}$ and $E = \bigcap_{r=1}^n E_r$, we have, since $\nu$ is the product measure on $K$ obtained from $(\nu_\lambda)_{\lambda \in \Lambda}$,

$$\mu(E) = \mu\left(\phi_S^{-1}\left(\prod_{\lambda \in \Lambda} B_\lambda\right)\right) = \prod_{\lambda \in \Lambda} \nu_\lambda(B_\lambda) = \prod_{r=1}^n \nu_{\lambda_r}(A_r) = \prod_{r=1}^n \mu(E_r),$$

so that (iii) is true.

Suppose finally that (i) is false. Then we can find $\Lambda_0 = (\lambda_1, \lambda_2, \cdots, \lambda_n) \subseteq \Lambda$, with $1 \leq n < \infty$, and integers $k_r$, for $1 \leq r \leq n$, such that $\prod_{r=1}^n g_{\lambda_r}^{k_r} = 1$, with $g_{\lambda_r}^{k_r} \neq 1$ for $r = 1, 2, \cdots, n$. This means that the character $f(\neq 1)$ of $K$ defined by $f(\omega) = \prod_{r=1}^n \omega_{\lambda_r}^{k_r}, \omega = (\omega_\lambda)_{\lambda \in \Lambda} \subseteq K$, is identically 1 on $\phi_S(H)$. But we can find $\omega \in K$ such that $f(\omega) \neq 1$, and then, by continuity of $f$, open sets $A_r \subseteq K_{\lambda_r}, 1 \leq r \leq n$, such that $f(\omega') \neq 1$ when $\omega' \in \prod_{\lambda \in \Lambda} B_\lambda$, the $B_\lambda$ being defined as before. Evidently $\phi_S^{-1}(\prod_{\lambda \in \Lambda} B_\lambda) = \emptyset$ and hence (again with the same notation) $E = \emptyset, \mu(E) = 0$. On the other hand

$$\prod_{r=1}^n \mu(E_r) = \prod_{r=1}^n \nu_{\lambda_r}(A_r) \neq 0,$$

and thus (iii) is false. Therefore statement (iii) implies (i), and the proof is complete.

I am indebted to Professor S. Kakutani for drawing my attention to Pontrjagin's proof of Kronecker's theorem. The foregoing proof
that statement (i) implies (ii) is essentially a rearrangement of part of Pontrjagin's argument (for which see [4, §37]).

**References**


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