A REMARK ON CURVATURE AND THE DIRICHLET PROBLEM

BY M. S. NARASIMHAN

Communicated by S. Bochner, August 17, 1959

1. Introduction. Let $M$ be a compact oriented Riemannian manifold with positive definite Ricci curvature. By a well-known theorem of Bochner-Myers [2, §26, p. 132] there are no nonzero harmonic forms of degree one on $M$. This result implies that "Dirichlet's problem" in the sense of §6 and §4 of reference [1] is solvable in $M$ and that there exists the Green's form of degree one on $M$ which is an elementary kernel for the Laplacian $\Delta$ (on 1-forms) [2, §31]. We shall point out in this note how, in this form, the result can be generalized to noncompact manifolds. We use the notations of [2].

2. Theorem. Let $M$ be an oriented $C^\infty$ Riemannian manifold countable at infinity. We assume that the mean curvature is positive and bounded away from zero, that is we assume that there exists a constant $C>0$ such that

$$R(v, v) \geq Cg(v, v)$$

for every tangent vector $v$, $R(v, v)$ denoting the Ricci form and $g(v, v)$, the metric form. Then Dirichlet's problem for 1-forms is solvable on $M$ and there exists the Green's form of degree one.

Proof. Referring to Proposition IV and §5 of [1], we have only to prove the Poincaré inequality for $C^\infty$ 1-forms with compact supports. Let $\alpha = (\alpha_1, \cdots, \alpha_h, \cdots, \alpha_n)$ be a $C^\infty$ 1-form with compact support. Then

$$(\Delta \alpha)_k = -\nabla^i \nabla_i \alpha_k - R^i_{kh} \alpha_k.$$

We have (see [2, p. 132]),

$$(\alpha, \Delta \alpha)_{L^2} = -\int \alpha^k \nabla_i \nabla_i \alpha_k * 1 - \int R^k_{kh} \alpha_k * 1$$

$$= \int \nabla_i \alpha_k \nabla_i \alpha_k * 1 - \int R_{kh} \alpha_k \alpha_k * 1,$$

using integration by parts. By assumption we have

$$-R_{kh} \alpha_k \alpha_k \geq Cg_{kh} \alpha_k \alpha_k$$

and we have

363
\[ \int \nabla_i \alpha^k \nabla_i \alpha_k \ast 1 \geq 0. \]

Consequently
\[ (\alpha, \Delta \alpha)_{L^2} \geq C \int g_{\alpha \beta} \alpha^\beta \ast 1 \]

that is,
\[ (d\alpha, d\alpha)_{L^2} + (\partial \alpha, \partial \alpha)_{L^2} \geq C(\alpha, \alpha)_{L^2} \]

which is Poincaré's inequality for 1-forms with compact supports.

**References**


**Tata Institute of Fundamental Research**, Bombay, and
**Centre National de la Recherche Scientifique**, Paris