there are numerous results and methods of intrinsic interest and importance.

The general viewpoint of the author's treatment is geometric as compared with the viewpoint of the transcendental theory of algebraic varieties.

STEWART S. CAIRNS

Trigonometric series. By A. Zygmund. 2d. ed., vols. I and II. New York, Cambridge University Press, 1959. 12+383 pp. and 7+354 pp. $15.00 each or $27.50 set.

In his course at the University of Cambridge, Professor Littlewood used to call the first edition of Zygmund's book "the Bible." This second edition, coming almost twenty-five years after the first one, will undoubtedly deserve this name even more, not only because it takes into account the work done in the field during this period, but also because the author, profiting from new experience and constant reflection on his past work, has introduced many topics which had been left aside in the first edition.

The book is dedicated to the memories of two polish mathematicians, A. Rajchman and J. Marcinkiewicz, who met both with a tragic end during the last world war: Rajchman was executed by the Nazis, while Marcinkiewicz died under circumstances not yet fully explained. The first one Zygmund calls "his teacher," the second one "his pupil," but both have considerably influenced the mathematical thought of Zygmund, who had an equal respect to the genius of these two mathematicians of the celebrated Polish school.

As the author states in the Preface, he has deliberately left aside all recent extensions of the theory to abstract fields and the second edition is, as the first one, devoted to the classical theory.

Chapter I contains the essential notions of analysis which shall be used throughout the book, such as: orthogonality, completeness and closure, Fourier-Stieltjes series, fundamental inequalities, convex functions, rearrangements and maximal theorems of Hardy and Littlewood. This is an improvement over the first edition, in which these fundamental notions were scattered through the book. It is only to be regretted, in the opinion of the reviewer, that the important theorems about abstract spaces and linear operations have been postponed until Chapter IV; they, like the other general theorems of analysis, belonged quite naturally to Chapter I, inasmuch as this chapter deals with convergence in the spaces L^r, with sets of first and second category, and with Baire's theorem.

Chapter II deals with the most elementary parts of the theory of

Fourier series: properties of the coefficients, elementary tests of convergence. It contains, as a new addition, the essential results about the theory of "smooth functions" which was discovered by the author in 1945 and whose applications do not seem yet to be exhausted.

Here again the notions of modulus of continuity and of smooth functions could have found their place in an introductory chapter, since they are used in analysis outside the field of trigonometric series. But one can well imagine that the author did not want to inflate too much the introduction, which could have made the reading of the book more difficult, if not to the specialist, perhaps to the student.

With *Chapter* III we enter the field of summability of Fourier series and of the conjugate series by the methods of arithmetic means, by Abel's method, and by Rogosinski's method (which has found its natural place here whereas in the first edition of the book it was in a quite different chapter). Moreover, a systematic treatment has been added of the approximation of continuous functions by trigonometric polynomials.

Chapter IV contains essentially the same topics as in the first edition, but the presentation has been greatly improved, and new material has been added. It begins with the fundamental theorem of Riesz and Fisher. It then proceeds to give an account of the theorems of Marcinkiewicz about integrals involving the function $\lambda(x)$, distance from a point x to a closed set. The use of this integral has proved very fruitful in many problems, and its extensions to higher dimensions have been of great value in the theory, by Calderon and Zygmund, of singular integrals and partial differential equations. The pure real variable proof of the existence of the conjugate function is also given by Marcinkiewicz's method.

We find there an exhaustive study of the functions of all classes L^r, in connection with their Fourier series, classical theorems about Fourier-Stieltjes series, and an improved version of the theorems of Orlicz about classes L_ϕ. In particular the author introduces a new norm for Orlicz spaces which gives satisfactory converses for Hölder's inequality.

Finally, the author has rightly introduced here the study of the majorants for Abel's and Cesaro's means, which in the first edition of the book were studied in a different chapter. As we have said before, general theorems about abstract spaces and linear operations are also to be found in this chapter.

Chapter V, as in the first edition, deals with special types of trigonometric series, e.g. series with monotonic coefficients, series of

Hardy and Littlewood, lacunary series, etc. New material consists, in particular, in the study of Fourier-Stieltjes series of continuous monotonic functions whose spectrum is a perfect set nowhere dense (as a special case Lebesgue functions constructed on sets of the Cantor type). New results have been added on lacunary series. Riesz products have been studied systematically. Finally, the study of Rademacher series is much more complete than in the first edition and contains all necessary tools for the study of trigonometric series whose coefficients have random signs.

Chapter VI is devoted to the study of absolute convergence of trigonometric series. Results found during the last twenty years have been added to the theory of sets of absolute convergence (the characterization of these sets in structural terms is a problem which is not yet solved). The rest of the chapter deals with the absolute convergence of Fourier series, and in particular with the Wiener-Lévy theorem, which is proved in a few lines by an extremely elegant method due to Calderon. The important discovery of Katzelson providing a sort of reciprocal of the theorem of Wiener-Lévy could not be included here, since it came after the publication of the book.

Chapter VII which contains the applications of the theory of functions of a complex variable to the theory of Fourier series has been improved mainly by addition of new results due to the introduction of smooth functions, and by the study of Nevanlinna's functions of the class N, which completes the theory of the functions of the classes H^p. Also the study of Blaschke's products is carried further than in the first edition. Finally some interesting topics about conformal mapping, in particular problems concerning the correspondence of boundaries, have been added at the end of the chapter.

Chapter VIII gives the classical examples, and some new ones, of continuous functions whose Fourier series diverges at a point, or in a set of points having the power of the continuum, also examples of continuous functions whose Fourier series do not converge uniformly. It then gives the example of Kolmogoroff of the Fourier series of a function of the class L (and even of the class H) which diverges almost everywhere. The construction of Kolmogoroff of an everywhere divergent Fourier series is also given and has been, as far as it is possible, simplified. The reader will also find here the more recent result of Marcinkiewicz about a function of the class L whose Fourier series has partial sums oscillating finitely at almost all points.

With *Chapter* IX we enter the field of Riemann's theory. We find there the classical results of La Vallée-Poussin, the theory of the sets of the type H, Rajchman's theory of formal multiplication, and theorems on localization. An account is given of the general theory of sets of uniqueness and sets of multiplicity, with the proof that sets H (introduced by Rajchman) and sets $H^{(n)}$ (recently discovered by Piatecki-Shapiro) are sets of uniqueness. A proof is given of the theorem of N. Bari, according to which the union of a denumerable infinity of closed sets of uniqueness is also a set of uniqueness. The chapter ends with the study of uniqueness of summable trigonometric series and of localization for trigonometric series whose coefficients do not tend to zero.

Chapter X begins the second volume and is devoted to an exhaustive theory of trigonometric interpolation. This is one of the essential additions to the first edition of the book, in which trigonometric interpolation was not dealt with. The reader will find there, in particular, interesting examples of divergence of interpolating polynomials.

Chapter XI contains the theory of generalized derivatives and its applications to differentiation of Fourier series, to necessary and sufficient conditions for summability C. The end of the chapter gives an account of the theorems in which Denjoy used his definition of the integral to show that an everywhere convergent trigonometric series is the Fourier series of its sum.

Chapter XII deals essentially with two topics: interpolation of linear operations, and additional properties of Fourier coefficients. The reader will find first the classical convexity theorem of Marcel Riesz about interpolation of linear operations, with the new proof given by Thorin, and with the applications to the Hausdorff-Young theorems. Next, the chapter deals with the more recent theory of interpolation in the classes H^p (instead of L^p), with the proof due to Calderon and Zygmund, and applications to the theorem of Hardy and Littlewood. It contains also the theorem discovered (but left unproved) by Marcinkiewicz about interpolation of quasi linear operations, the proof of which has been given very recently by Zygmund.

We find then the proof of Paley's theorems on Fourier coefficients of general orthogonal systems, and the theorems of Hardy and Littlewood about rearrangement of Fourier coefficients.

The rest of the chapter deals with lacunary Fourier coefficients, with theory of fractional integration and its applications to Fourier series, and finally with theorens about Fourier-Stieltjes coefficients, where special emphasis is put on the problem of the behavior of Fourier-Stieltjes coefficients of singular functions. Among these functions, the Cantor-Lebesgue functions having for spectrum a perfect set of the Cantor type and of constant ratio of dissection are studied in detail, and the corresponding theorems of arithmetical type are given. Finally the reader will find at the end of the chapter the complete recent solution of the classification of sets of the Cantor type in sets of uniqueness and sets of multiplicity, when the ratio of dissection is constant.

Chapter XIII contains the theory of the order of magnitude of the the partial sums of Fourier series, considered almost everywhere. It starts with the theorem of Kolmogoroff and Seliverstoff for functions of the class L^2, and contains a very interesting and hitherto unpublished result of Calderon, according to which, in order to prove the existence of a $f \in L^2$ with Fourier series divergent almost everywhere, it is enough to prove the unboundedness of the integral of Kolmogoroff and Seliverstoff. The case $f \in L^p$ $(1 < p \leqq 2)$ is then considered. The chapter contains the interesting test of Marcinkiewicz about convergence at almost all points of a set, the study of majorants and minorants of the partial sums, and theorems on partial sums of power series.

The same chapter contains the theory of strong summability, the theorems of Mencheff-Rademacher on convergence almost everywhere for general systems of orthogonal functions, and the relation, first studied by Beurling, between capacity of sets and convergence of Fourier series.

In *Chapter* XIV we find material which, quite independently of its applications to trigonometric series, has deep function-theoretic interest in itself. It contains the study of boundary behavior of analytic and harmonic functions. It begins with the classical theorem of Privaloff, of which, in addition to the usual proof, the recent proof of Calderon, independent of the theory of conformal mapping, is also given. The new results of Rudin, Bagemihl and Seidel are quoted without proof. The chapter contains also the study of the function $s(\theta)$ of Lusin and its applications, of the function $g(\theta)$ of Littlewood-Paley, and of the analogous function $\mu(\theta)$ of Marcinkiewicz.

Both Chapters XIII and XIV contain the theorems about sets of convergence of conjugate series.

Chapter XV is entirely devoted to the application to Fourier series of the function $g(\theta)$ of Littlewood and Paley studied in the preceding chapter. It gives the results of Littlewood and Paley themselves, as well as extensions of Marcinkiewicz, Zygmund and others.

Chapter XVI contains a number of topics about Fourier integrals. First of all the classical theorems about convergence and summability. Secondly the theory of Fourier transforms with Plancherel's theorems, transforms on the classes L^p $(1 < p \leqq 2)$, Hilbert transforms.

We then find a section about Fourier-Stieltjes transforms with applications to probability calculus, and to certain recent theorems about lacunary trigonometric series and trigonometric series whose terms have random signs.

The chapter contains several important additions to the corresponding chapter of the first edition. The theorem of Paley-Wiener is given.

The end contains Riemann's theory of trigonometric integrals, equiconvergence theorems, and problems of uniqueness.

The reviewer regrets that no account is given of Wiener's Tauberian theorem which belongs properly to the theory of Fourier integrals, and which a serious student might wish to learn as the starting point of modern harmonic analysis.

Chapter XVII deals with multiple Fourier series. It does not pretend to give a complete account of the theory of Fourier series in several variables, but is mainly concerned with the theory of rectangular summability. "Restricted summability" at almost every point is proved. In a second part of the chapter the reader will find interesting results, due to Zygmund and to Calderon, about power series of several complex variables, and particularly about the existence, under certain remarkable conditions, of nontangential limits.

This necessarily dry enumeration of the topics contained in the book could hardly give a right impression of its richness. Each chapter is completed by a series of supplementary theorems which could not find their place in the text, and which, when not too difficult, provide good exercises for the student. Besides, the book ends with notes on each chapter which have more than historical interest.

Since one tends always to be more exacting when one is put in front of an almost perfect work, the reviewer may venture to regret the absence in the book of two topics. First, one could have liked to find an account of Fourier integrals of distributions in the sense of Schwartz, which are now used by so many authors. Secondly, a few indications about the theory of spectral analysis and spectral synthesis would not have been out of place, although one might rightly say that they belong more to a general treatise of harmonic analysis rather than to a more particular treatise on trigonometric series. But the points of contact are numerous and while the long sought result of Malliavin on the impossibility of spectral synthesis of bounded functions on the line could not have been included in the book, which appeared before Malliavin's notes, positive results like the ones of Herz could have been of interest even for the student of trigonometric series. Likewise, the reader could have liked to learn more details about the sets introduced by Carleson and Helson which, on account of the questions remaining still open, raise interesting problems for the research student.

RAPHAEL SALEM

Tauberian theorems. By Harry Raymond Pitt. Tata Institute of Fundamental Research Monographs on Mathematics and Physics, vol. 2, Oxford University Press, 1958. 10+174 pp. 30 sh.

The terminology used in this book is well known. A theorem which asserts that, for the transformation

$$g(u) = \int_{-\infty}^{+\infty} k(u, v)s(v)dv, \tag{1}$$

$s(v) \to a$ as $v \to +\infty$ implies $g(u) \to a$ as $u \to +\infty$ is an *Abelian* or a direct theorem. A *Tauberian* or an inverse theorem asserts that conversely if $g(u) \to a$ and if $s(v)$ satisfies some additional condition, the so called Tauberian condition, often of the type that $s(v)$ changes slowly with v, then $s(v) \to a$. A *Mercerian* theorem is an inverse theorem which holds without a Tauberian condition. A theorem is called *special* if it refers to a specific kernel k, and *general* if it holds for an extensive class of kernels.

Tauberian theorems obtained by N. Wiener some 25 years ago, and subsequent contributions of the author, play a central rôle in the whole theory. They are the main subject of this book. Accordingly, readers will find, for example, little about estimation of Tauberian constants, about Tauberian theorems of function theoretic type, asymptotic theorems, best Tauberian conditions and about application of Banach algebras or of locally convex spaces.

The content is as follows: Chapter I–III contain a discussion of very general Tauberian conditions, of slowly decreasing functions; this is followed by elementary general Tauberian theorems and theorems in which boundedness of $g(u)$ implies that of $s(v)$. Special Tauberian theorems are given for the methods of Cesàro, Riesz, Abel and Borel; in the last two cases the proofs furnish also the corresponding high-indices theorems.

Chapters IV and V constitute the main part of the book. After theorems from harmonic analysis about the properties of an analytic function of the Fourier transform $K(t)$ of $k(t)$, the main theorem is proved: if the kernel of (1) is $k(u-v)$, if $K(t) \neq 0$ and if $s(v)$ is bounded, then $S \leqq \epsilon + C(\epsilon)G$, where $S = \lim\sup |s(v)|$, $G = \lim\sup |g(u)|$. Other classical Wiener theorems follow easily. Refinements of these are then discussed: Tauberian theorems where $K(t) \neq 0$ is assumed