RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

BEURLING’S TEST FOR ABSOLUTE CONVERGENCE OF FOURIER SERIES

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We say, with Beurling [1], that \( f \) is a contraction of \( g \) if \( |f(x) - f(y)| \leq |g(x) - g(y)| \). Beurling established the following theorem.

**Theorem 1.** If \( f \) and \( g \) are continuous even functions, of period \( 2\pi \), with Fourier cosine coefficients \( c_n \) and \( g_n \), if \( f \) is a contraction of \( g \), and if \( |g_n| \leq \gamma_n \), where \( \gamma_n \downarrow 0 \) and \( \sum \gamma_n < \infty \); then \( \sum |c_n| < \infty \).

Since saying that \( f \) is a contraction of \( g \) is essentially the same as saying that \( f \) is a Lipschitzian function of \( g \), theorems like Theorem 1 have gained in interest since the recent discovery [3] that in general only analytic functions operate on all absolutely convergent Fourier series with preservation of absolute convergence.

I shall give an elementary proof of a generalization of Theorem 1. Further investigations along these lines by M. Kinukawa, M. and S. Izumi, and the author are in progress.

**Theorem 2.** Theorem 1 remains true when the hypothesis that \( \gamma_n \downarrow 0 \) is replaced by

\[
\sum_{n=1}^{\infty} n^{-9/2} \left\{ \sum_{k=1}^{n} k^{2} \gamma_k \right\}^{1/2} + \sum_{n=1}^{\infty} x_{n}^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k \right\}^{1/2} < \infty.
\]

Beurling’s theorem follows from Theorem 2 since when \( \gamma_n \downarrow 0 \), or even when \( n^{-\lambda} \gamma_n \downarrow 0 (\lambda > 0) \), the series in (1) converge if \( \sum \gamma_n \) converges. This lemma has been proved by Konyushkov [4]; it is a corollary of more general results that I discuss elsewhere [2] by a different method; a proof by a still different method, that yields exact constants in the inequalities involved, has been obtained by S. Łojasiewicz.

Condition (1) can also be satisfied when \( \{\gamma_n\} \) is not required to satisfy a condition of monotonicity. If \( \sum n^{1/2} \gamma_n < \infty \), the left-hand side of (1) does not exceed, by Jensen’s inequality,
\[
\sum_{n=1}^{\infty} n^{-3/2} \sum_{k \geq n} k \gamma_k + \sum_{n=1}^{\infty} n^{-1/2} \sum_{k > n} \gamma_k = \sum_{k=1}^{\infty} k \gamma_k \sum_{n \geq k} n^{-3/2} + \sum_{n=1}^{\infty} \gamma_k \sum_{n < k} n^{-1/2} \leq \sum_{k=1}^{\infty} \gamma_k O(k^{1/2}).
\]

Hence the hypothesis \( \gamma_n \downarrow 0 \) in Theorem 1 can be replaced by \( \sum n^{1/2} \gamma_n < \infty \). In other words, the cosine series of \( f \) converges absolutely if \( f \) is a contraction of \( g \) and \( g \) has a derivative of order 1/2 that has an absolutely convergent Fourier series.

**Proof of Theorem 2.** What is actually used in Beurling's proof (and in this one) is the condition

\[(2) \quad \int_0^\pi |f(x + \delta) - f(x)|^2 dx \leq \int_0^\pi |g(x + \delta) - g(x)|^2 dx,
\]

which may be thought of as saying that \( f \) is an average contraction of \( g \). Take \( \delta = \pi/n \), where \( n \) is a positive integer. By Parseval's theorem we can write (2) in the form

\[(3) \quad \sum_{k=1}^{\infty} c_k^2 \sin^2 \left( \frac{k \pi}{2n} \right) \leq \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 \left( \frac{k \pi}{2n} \right) \leq \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 \left( \frac{k \pi}{2n} \right).
\]

Let \( \phi_n = \sum_{k=1}^{n} k |c_k| \). Then

\[\phi_n \leq n^{1/2} \left\{ \sum_{k=1}^{n} k \gamma_k \right\}^{1/2},
\]

and by partial summation

\[
\sum_{n=1}^{N} |c_n| = \sum_{n=1}^{N} n^{-1}(\phi_n - \phi_{n-1}) = \sum_{n=1}^{N-1} \phi_n(n^{-1} - (n + 1)^{-1}) + \phi_N/N \leq \sum_{n=1}^{N} \phi_n/n^2 + \phi_N/N \leq \sum_{n=1}^{N} n^{-3/2} \left\{ \sum_{k=1}^{n} k \gamma_k \right\}^{1/2} + N^{-1/2} \left\{ \sum_{k=1}^{N} k \gamma_k \right\}^{1/2} = S_1 + S_2,
\]
say. From (3) we have, since \( \sin x \geq 2x/\pi \) for \( 0 \leq x \leq \pi/2 \),
\[
\sum_{k=1}^{n} k^2 c_k^2 \leq n \sum_{k=1}^{n} c_k^2 \sin^2 \left( \frac{k\pi}{2n} \right) \leq n \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 \left( \frac{k\pi}{2n} \right),
\]
and hence
\[
S_1 \leq \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 \left( \frac{k\pi}{2n} \right) \right\}^{1/2},
\]
(4)
\[
S_2 \leq N^{1/2} \left\{ \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 \left( \frac{k\pi}{2n} \right) \right\}^{1/2}.
\]
(5)
We must now show that \( S_1 \) and \( S_2 \) are bounded as \( N \to \infty \). For \( S_2 \), we have
\[
S_2^2 \leq N \sum_{k=1}^{N} k^2 \gamma_k^2 \pi^2 / (4N^2) + N \sum_{k=N+1}^{\infty} \gamma_k^2 \leq \frac{1}{4} \pi^2 N^{-1} \sum_{k=1}^{N} k^2 \gamma_k^2 + N \sum_{k=N+1}^{\infty} \gamma_k^2 = T_1 + T_2.
\]
The second series in (1) has decreasing terms, which must therefore be \( O(1/n) \); hence \( T_2 = O(1) \).
Call the first series in (1) \( \sum n^{-3/2}A_n \); here \( A_n \) increases. We have
\[
\sum_{N}^{2N} n^{-3/2}A_n \geq A_N \sum_{N}^{2N} n^{-3/2} \geq C A_N N^{-1/2},
\]
with an irrelevant constant \( C \), and hence \( A_N^2/N \to 0 \). Thus, in particular, \( T_1 = O(1) \). This disposes of \( S_2 \).
We write
\[
S_1 \leq \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=1}^{n} + \sum_{k=n+1}^{\infty} \right\}^{1/2}
\leq \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=1}^{n} \gamma_k^2 \sin^2 \left( \frac{k\pi}{2n} \right) \right\}^{1/2} + \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \right\}^{1/2}
\leq (\pi/2) \sum_{n=1}^{N} n^{-3/2} \left\{ \sum_{k=1}^{n} \gamma_k^2 \right\}^{1/2} + \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \right\}^{1/2}.
\]
If we assume (1), the two sums on the right are bounded. This completes the proof.

REFERENCES


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