THE EQUIVALENCE OF FIBER SPACES AND BUNDLES

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Communicated by Hans Samelson, October 8, 1959

1. Introduction. The objective of this paper is to verify the conjecture made in [2] that every Hurewicz fibration [3] over a polyhedral base is fiber homotopy equivalent to a Steenrod fiber bundle [6]. The result relies heavily on Milnor's universal bundle construction [4] and the following extension [2] of a theorem of A. Dold [1].

THEOREM. If \{E_1, p_1, X\} and \{E_2, p_2, X\} are Hurewicz fibrations over a connected CW-complex X and if f: E_1 \rightarrow E_2 is a fiber-preserving map such that f restricted to some fiber is a homotopy equivalence, then f is a fiber homotopy equivalence.

2. The associated bundle. Let \( \pi: E \rightarrow X \) denote a map, where X is a connected, locally finite polyhedron. Furthermore following Milnor's notation in [4], let \( \mathcal{S}, \mathcal{E}, \mathcal{G} \) denote, respectively, the simplicial paths in X, the simplicial paths emanating from a fixed vertex \( v_0 \) and the simplicial loops at \( v_0 \). If \( \alpha = [x_n, \cdots, x_0] \) is a simplicial path in X we will find it convenient to set \( \alpha(0) = x_0, \alpha(1) = x_n \). Now, define

\[ \Omega_\alpha = \{(e, \alpha) \in E \times \mathcal{S} \mid \pi(e) = \alpha(0)\} \]

and a map \( \xi: \Omega_\alpha \rightarrow X \) by

\[ \xi(e, \alpha) = \alpha(1). \]

Furthermore, let

\[ A = \xi^{-1}(v_0) = \{(e, \alpha) \mid \pi(e) = \alpha(0), \alpha(1) = v_0\}. \]

LEMA. \( \{\Omega_\alpha, \xi, X, A, \mathcal{G}\} \) is a Steenrod fiber bundle.

PROOF. Since the proof is entirely analogous to Milnor's proof [4] that \( \mathcal{E} \) is a bundle over X, we content ourselves with a brief outline. The action \( \mu: \mathcal{G} \times A \rightarrow A \) is defined as follows:

\[ \mu(g, (e, \alpha)) = (e, g\alpha). \]

Now, let \( v_j \) denote a vertex in X and \( V_j \) the star neighborhood of \( v_j \). The coordinate functions

\[ \phi_j: V_j \times A \rightarrow \xi^{-1}(V_j) \]

are defined by

\[ \phi_j = \xi^{-1}(\xi(v_j)). \]
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\[ \phi_j(x, (e, \alpha)) = (e, [x, v_j] e \alpha) \]

where \( e_j \) is a fixed simplicial path from \( v_0 \) to \( v_j \). We leave the remaining details to the reader.

Now, define \( f: E \to \Omega_x \) by

\[ f(e) = (e, [\pi(e), \pi(e)]) \]

The following diagram is easily seen commutative:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & \Omega_x \\
\pi \downarrow & \nearrow \xi & \\
X & & \\
\end{array}
\]

3. The equivalence theorem. Let \( \pi: E \to X, f: E \to \Omega_x \) be as in §2.

**Theorem.** If \( \{E, \pi, X\} \) is a Hurewicz fibration, then \( f \) is a fiber homotopy equivalence.

**Proof.** Let \( F = \pi^{-1}(v_0) \) denote the fiber in \( E \) over \( v_0 \). Then, in view of the theorem mentioned in the introduction, it suffices to show that \( f' = f| F: F \to A \) is a homotopy equivalence.

Let \( \overline{X} \) denote the space of ordinary paths in \( X \) ending at \( v_0 \) and let \( \eta: \overline{X} \to X \) denote the fiber map given by \( \eta(\alpha) = \alpha(0) \). Furthermore, let \( \overline{E} \) denote the space of simplicial paths \( [x_n, \ldots, x_0] \) on \( X \) such that \( x_n = v_0 \). Then, since \( \overline{E} \) is homeomorphic to \( E \) under the correspondence \( \overline{[x_n, \ldots, x_0]} \leftrightarrow [x_n, \ldots, x_0] \), \( \overline{E} \) is a fiber bundle over \( X \) with fiber map \( p: \overline{E} \to X \), given by \( p([x_n, \ldots, x_0]) = x_0 \) and fiber \( \mathcal{G} \).

Consider then the fiber-preserving map \( \overline{h} \)

\[
\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{h}} & \overline{E} \\
\eta \downarrow & \nearrow \eta & \\
X & & \\
\end{array}
\]

defined as follows: Let \( \lambda \) denote a regular lifting function for \( \{\overline{E}, p, X\} \) and if \( \alpha \in \overline{X} \), set \( \lambda(t) = \alpha(1) - t \), \( 0 \leq t \leq 1 \). Finally, define

\[ \overline{h}(\alpha) = \lambda([v_0, v_0], \alpha)(1) \]

Now, \( \overline{X} \) and \( \overline{E} \) are contractible, \( \eta^{-1}(v_0) = \Omega(X) \), the space of ordinary loops on \( X \), is dominated by a CW-complex and \( \mathcal{G} \) is a CW-complex. Therefore \( \overline{h} \) restricted to \( \Omega(X) \) is a homotopy equivalence and we may conclude that \( \overline{h} \) is a fiber homotopy equivalence. Thus \( \overline{h} \) possesses a fiber homotopy inverse \( h \). If \( \bar{v}_0 \in \overline{X} \) is the constant path and \( [v_0, v_0] \in \mathcal{G} \) is the identity in \( \mathcal{G} \), then \( \overline{h}(\bar{v}_0) = [v_0, v_0] \) and \( h \) may be
chosen so that \( h([v_0, v_0]) = \bar{v}_0 \). We employ \( h \) and \( \bar{h} \) to define an auxiliary map \( \chi: A \to A \) as follows. Define

\[
\chi(e, \alpha) = (e, \bar{h}h(\alpha)).
\]

Since \( \bar{h}h \) is fiber homotopic to the identity map \( E \to E \), \( \chi \sim 1: A \to A \).

Next, we define a homotopy \( H: A \times I \to A \). If \( \omega \) is an ordinary path in \( X \) and \( 0 \leq s, t \leq 1 \), set

\[
\omega_s(t) = \omega(st)
\]

and

\[
\omega^*(t) = \omega(s + t - st).
\]

Then, define, for \( 0 \leq s \leq 1 \),

\[
H((e, \alpha), s) = \{ \lambda(e, \[h(\alpha)\](s)), \bar{h}(\[h(\alpha)\]^s)(0) \}
\]

where \( \lambda \) is a regular lifting function for \( \{E, \pi, B\} \). Note that \( \pi \lambda(e, [h(\alpha)])(1) = h(\alpha)(s) = \bar{h}(\[h(\alpha)\]^s)(0) \) since \( \bar{h} \) preserves end points and \( \[h(\alpha)\]^s(0) = h(\alpha)(s) \). Also \( \bar{h}(\[h(\alpha)\]^s)(1) = v_0 \) for the same reason. Thus, \( H((e, \alpha), s) \subseteq A \). Furthermore,

\[
H_0(e, \alpha) = (e, \bar{h}h(\alpha)) = \chi(e, \alpha),
\]

\[
H_1(e, \alpha) = \{ \lambda(e, h(\alpha))(1), [v_0, v_0] \}
\]

where \( [v_0, v_0] \) is the identity in \( \bar{G} \).

Finally, we define the required homotopy inverse for \( f': F \to A \). Set

\[
g(e, \alpha) = \lambda(e, h(\alpha))(1).
\]

Then, if \( y \in F \),

\[
gf'(y) = g(y, [v_0, v_0]) = \lambda(y, \bar{v}_0)(1) = y
\]

and hence \( gf' = 1 \). Also, if \( (e, \alpha) \in A \),

\[
f'g(e, \alpha) = (\lambda(e, h(\alpha))(1), [v_0, v_0]) = H_1(e, \alpha).
\]

Therefore \( f'g \sim \chi \sim 1 \) and \( g \) is a homotopy inverse for \( f' \). This proves the equivalence theorem.

**Remark.** It is not difficult to check that \( F \) considered as a subset of \( A \) is actually a strong deformation retract of \( A \).

**Remark.** It is quite clear that our main result is false for Serre fibrations [5] since there exist Serre fibrations over the unit interval whose fibers are not of the same homotopy type. Also, it is possible to exhibit examples of Hurewicz fibrations with 0-connected but not locally contractible base spaces for which our main result is false.
4. Extensions. The Equivalence Theorem is also valid if the base space $X$ is dominated by a locally finite polyhedron. Thus, our main result can be stated as follows.

**Theorem.** Every Hurewicz fibration over a base space dominated by a locally finite polyhedron is fiber homotopy equivalent to a Steenrod fiber bundle.

An interesting application is the following corollary.

**Corollary.** If $X$ is a connected space dominated by a locally finite polyhedron, then for every integer $n \geq 1$, there exist $n$-connective Steenrod fiber bundles over $X$.

**Proof.** One merely applies the above theorem to the $n$-connective Hurewicz fibrations over $X$ given by G. W. Whitehead in [7].

**Bibliography**