RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A CANONICAL FORM FOR AN ANALYTIC FUNCTION OF SEVERAL VARIABLES AT A CRITICAL POINT

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Theorem. Let $f(z, w)$ be analytic in $(z, w)$ for small $|z|$ and $|w|$. Let $n > 1$, (since the case $n = 1$ is trivial), let

$$\frac{\partial^k f}{\partial w^k}(0, 0) = 0 \quad 1 \leq k < n,$$

and let

$$\frac{\partial^n f}{\partial w^n}(0, 0) \neq 0.$$

Then there is an analytic function $g$ of $(z, s)$ for small $|z|$ and $|s|$ such that setting

$$w = s + s^n g(z, s)$$

in $f(z, w)$ yields $f(z, w) = P(z, s)$ where $P$ is a polynomial in $s$

$$f(z, w) = P(z, s) = \sum_{i=0}^{n} p_j(z) s^j.$$

The $p_j$ are analytic for small $|z|$,

$$p_j(0) = 0 \quad 1 \leq j \leq n - 1,$$

and $p_n(0) \neq 0$. Clearly of course (1) implies $s = w + w^n h(z, w)$ where $h$ is analytic for small $|z|$ and $|w|$. Thus for any small $z$ there is a one to one analytic correspondence between $w$ and $s$ for small $|w|$ and $|s|$.

The result (2) is somewhat reminiscent of the Weierstrass preparation theorem but is different in that the polynomial on the right of (2) is not multiplied by a function of $(z, s)$. On the other hand to achieve the canonical polynomial (2), it is necessary to use the change of variables (1).

A case of this theorem where $n = 3$ arises in the transformation of confluent saddle points to Airy integrals [1; 2] and is treated there.

An indication of the proof of the theorem follows. Since one can take $p_0(z)$ in (2) as $f(z, 0)$ there is no loss of generality in treating the

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case where \( f(z, 0) = 0 \). There is also no restriction in assuming that 
\( \partial^n f/\partial w^n(0, 0) = n! \). Hence it can be assumed that

\[
(3) \quad f(z, w) = z^a \sum_{1}^{n} b_j(z)w^j + w^n + w^{n+1}f(z, w).
\]

Here \( a > 0 \) is an integer and \( \phi \) is analytic in \((z, w)\).

If we now introduce

\[
(4) \quad w_1 = w[1 + w\phi(z, w)]^{1/n}
\]

then for small \(|z|\) and \(|w|\)

\[
w_1 = w + c_2(z)w^2 + c_3(z)w^3 + \cdots,
\]

and there is an inverse

\[
(5) \quad w = w_1 + w_2\psi(z, w_1).
\]

Using (4) and (5) in (3), \( f(z, w) \) becomes

\[
f_1(z, w_1) = z^a \sum_{1}^{n} b_j(z)[w_1 + w_2\psi(z, w)]^j + w^n,
\]

or

\[
(6) \quad f_1(z, w) = z^a \sum_{1}^{n} b^{(1)}_j(z)w_1^j + w_1^n + z^a w_1^{n+1}f_1(z, w_1),
\]

where \( \phi_1(z, w_1) \) is analytic for small \(|z|\) and \(|w_1| \leq \rho_1\). The proof consists in iterating (4) and showing convergence. Thus the next step involves setting

\[
w_2 = w_1[1 + w_1\phi_1(z, w_1)]^{1/n}
\]

in (6), giving \( f_1 = f_2 \) where

\[
f_2(z, w_2) = z^a \sum_{1}^{n} b^{(2)}_j(z)w_2^j + w_2^n + z^a w_2^{n+1}\phi_2(z, w_2),
\]

The important point is that at each stage \( \phi_n \) comes from terms in \( f_{n-1} \) multiplied by \( z^a \). It can be shown that \( r_1 > 0 \) can be chosen so that if \(|z| < r_1\), the occurrence of \(|z|^a \leq r_1^a \) in \( \phi_n \) at each stage causes convergence finally for \(|z| < r_1\) and \(|w_1| < \rho_0/2 \) where

\[
\rho_0 = \rho_1(1 - 1/2)(1 - 1/4)(1 - 1/8) \cdots.
\]

A detailed proof will be given elsewhere. The case where \( z \) is replaced by several variables is treated in much the same way.

**References**


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