SOME PROPERTIES OF CHARACTERS OF FINITE SOLVABLE GROUPS

BY PAUL FONG

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The purpose of this note is to announce two results on properties of the characters of a finite solvable group $G$. The first is a necessary and sufficient condition on the irreducible characters of $G$ in order that its Sylow subgroups be abelian. The second is a necessary condition on the irreducible characters of $G$ in order that certain $p$-subgroups of $G$ be abelian. These results confirm certain assertions in a conjecture by Brauer in [1]. In order to formulate the results, some definitions from modular representation theory are needed. For the motivation behind these definitions, see [2].

Let $G$ be a finite group of order $g$, $F$ an algebraic number field of finite degree such that the irreducible representations of $G$ in $F$ are absolutely irreducible. If $p$ is a fixed rational prime, there is a grouping of the absolutely irreducible characters $\chi_1, \chi_2, \cdots, \chi_k$ of $G$ into disjoint sets called the blocks of $G$ (for the prime $p$). This grouping can be made in the following way: let $\mathfrak{p}$ be a prime ideal divisor of $p$ in $F$. Then two irreducible characters $\chi_i$ and $\chi_j$ are in the same block $B_t$ of $G$ if and only if

$$\frac{g}{n(\sigma)} \chi_i(\sigma) \equiv \frac{g}{n(\sigma)} \chi_j(\sigma) \pmod{b}$$

for all $\sigma$ in $G$. Here $n(\sigma)$ is the order of the normalizer of $\sigma$ in $G$, $x_i$ the degree of $\chi_i$, $x_j$ the degree of $\chi_j$. Since the numbers occurring in the above congruence can be computed from the character table of $G$, the blocks of $G$ can therefore be determined once the character table of $G$ is known.

**Theorem 1.** Let $G$ be a finite solvable group, $B_1$ the block containing the principal or 1-character of $G$. Then a necessary and sufficient condition for the Sylow $p$-subgroups of $G$ to be abelian is that every character in $B_1$ has degree relatively prime to $p$.

Since our two results are related, we shall state the second result before indicating their proofs. Let $p^a$ be the highest power of $p$ dividing the order of the finite group $G$ ($G$ not necessarily solvable). The defect of a block $B_t$ of $G$ is the smallest non-negative integer $d$ such that $p^{a-d}$ divides the degree $x_i$ of every character $\chi_i$ in $B_t$. If the exact power of $p$ dividing the degree of a character $\chi_i$ in $B_t$ is $p^{a-d+e}$, where $e \geq 0$, we define the height of $\chi_i$ to be $e$. In [2] there is associated to
each block $B_i$ its defect group $D$, a $p$-subgroup of $G$ determined uniquely up to conjugate subgroups in $G$. $D$ has order $p^d$, where $d$ is the defect of $B_i$. The conjecture by Brauer is that the defect group $D$ of a block $B_i$ is abelian if and only if every character in $B_i$ has height 0. Theorem 1 is a special case of the conjecture, since the defect groups of $B_1$ are the Sylow $p$-subgroups of $G$.

**Theorem 2.** Let $G$ be a finite solvable group, $B_i$ a block of $G$ with $D$ as defect group. If the center of $D$ has index $p^c$ in $D$, then every character in $B_i$ has height less than or equal to $c$. In particular, if $D$ is abelian, then every character in $B_i$ has height 0.

The necessity of the condition in Theorem 1 is included in Theorem 2. The proofs of both theorems use induction on the order of $G$. In order to carry through the induction, the following lemma is needed: let $H$ be a normal subgroup of prime index of a group $G$, $G$ not necessarily solvable. If $\chi_i$ is an irreducible character of $G$ in a block with $D$ as defect group, then some irreducible constituent of the restriction $\chi_i|_H$ of $\chi_i$ to $H$ is in a block of $H$ with $D\cap H$ as defect group. Here $D\cap H\neq D$ only in the case $(G:H) = p$ and the irreducible constituents of $\chi_i|_H$ are in one block of $H$. Theorem 2 follows by a direct application of this lemma except in the case where $D\cap H\neq D$ and $\chi_i|_H$ is reducible. However, it is then possible to show that the center of $D$ lies in $H$ and induction will work.

The proof of the sufficiency of the condition in Theorem 1 is rather lengthy. The bare outline of the proof is as follows. By induction we can reduce the problem first to the case where every maximal normal subgroup of $G$ has index $p$, secondly to the case where every minimal normal subgroup of $G$ has order a $p$-power. These restrictions on the maximal and minimal normal subgroups of $G$ imply that the minimal normal subgroups of $G$ have order $p$. If Theorem 1 were not true, it would then be possible to construct an irreducible character of a suitable subgroup $M$ of $G$ for which Theorem 2 would be false. The subgroup $M$ is the normalizer in $G$ of a fixed Sylow $p$-subgroup of $H$, where $H$ is a maximal normal subgroup of $G$.

**References**


Harvard University