GROUPS OF AUTOMORPHISMS OF ALMOST KAHLER MANIFOLDS

BY S. I. GOLDBERG

Communicated by S. Bochner, January 9, 1960

1. Let $M$ be a compact almost Kaehler manifold of real dimension $2n$. The fundamental 2-form $\omega$ (which together with the metric $g$ of $M$ defines the almost Kaehlerian structure) is harmonic and therefore invariant by every infinitesimal isometry [2]. Let $X$ be an infinitesimal conformal transformation of $M$. Then, for all $n>1$ we shall show that $X$ is in fact an infinitesimal isometry. Indeed, the following theorem is proved:

**Theorem 1.** The largest connected Lie group of conformal transformations of a compact almost Kaehler manifold $M^{2n}$ ($n>1$) coincides with the largest connected group of automorphisms of the almost Kaehlerian structure. Moreover, the infinitesimal automorphisms are infinitesimal isometries.

This generalizes a previous result [3]. If the almost complex structure is completely integrable and comes from a complex analytic structure we obtain the theorem of Lichnerowicz [5] whose methods it seems cannot be extended to include the almost Kaehler manifolds.

In the noncompact case if we consider infinitesimal conformal maps whose covariant forms are closed, a much wider class of manifolds may be considered.

2. Let $X$ be a vector field on $M$ whose image by the almost complex structure operator $J$ is “closed,” that is, its covariant form $C\xi$ is closed where $C$ is the complex structure operator applied to forms. Then, $X$ is an infinitesimal automorphism of $M$. Denote by $t(X)$ the tensorfield $\theta(X)J$ modulo $i(X)D\omega$ where $\theta(X)$, $i(X)$ and $D$ are the Lie derivative, interior product and covariant differential operators, resp. For Kaehler manifolds $D\omega$ vanishes, and so $t(X)$ and $\theta(X)J$ coincide. In this case, the vanishing of $t(X)$ characterizes the infinitesimal analytic transformations. Let $t$ be a covariant real tensor of order 2 and denote by $J$ again the operator

---

1 This research was supported by the United States Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF49(638)-14.

2 The manifolds, differential forms and tensor fields considered are assumed to be of class $C^\infty$. 
Clearly, $J\omega = g$. For any vector field $X$ on a Kaehler manifold it can be shown that

\[ \bar{\theta}(X)\omega - \theta(X)\omega = \delta \xi \cdot \omega - 2\bar{\Pi}(X) \]

where $\bar{\theta}(X)$ denotes the dual of $\theta(X)$ and $\bar{\Pi}(X)$ denotes the 2-form corresponding to the skew-symmetric part of $\bar{t}(X)$ \cite{3}.

**Lemma 1.** For any vector field $X$ on a Kaehler manifold $M$

\[ ||\theta(X)\omega||^2 = ||\delta \xi||^2 + 2(\bar{\Pi}(X), \theta(X)\omega) \]

where $||\alpha||^2 = (\alpha, \alpha) = \int_M \alpha \wedge \ast \alpha$ for any $p$-form $\alpha$.

**Theorem 2.** A vector field $X$ defines an infinitesimal analytic transformation of a Kaehler manifold if and only if $J\theta(X)\omega = \theta(X)g$, that is when applied to $\omega$ the operators $\theta(X)$ and $J$ commute.

This follows from the fact that $t(X) = \theta(X)\omega + J\theta(X)g$ a relation used in establishing formula (1). Lemma 1 therefore implies the following

**Corollary.** For an infinitesimal analytic transformation

\[ ||\theta(X)\omega|| = ||\delta \xi||. \]

Hence, a divergence free analytic map is an infinitesimal automorphism of the Kaehler structure.

**Lemma 2.** For an infinitesimal conformal transformation $X$ of a Kaehler manifold

\[ t(X) = \theta(X)\omega + \frac{1}{n} \delta \xi \cdot \omega. \]

Our notation does not distinguish between the 2-form $t(X)$ and the corresponding tensorfield. If $\xi$ is closed, $t(X)$ is symmetric and must therefore vanish. We conclude from the lemma that $d\delta \xi = 0$ for $n > 1$, that is $X$ is homothetic. Since a homothetic map of a complete Riemannian manifold which is not locally flat is isometric we conclude

**Theorem 3.** A closed infinitesimal conformal transformation of a complete Kaehler manifold $M^{2n}$ ($n > 1$) which is not locally flat is an automorphism of the Kaehler structure.

In the locally flat case an infinitesimal affine transformation $X$ is isometric if and only if its length is bounded, that is the vector field on $M$ defining $X$ has bounded length. Hence, since a homothetic map is affine we have
Theorem 4. A closed infinitesimal conformal map of a complete locally flat Kaehler manifold $M^{2n}$ ($n > 1$) is an automorphism of the Kaehler structure if and only if its length is bounded.

Remarks. (a) Every conformal map of a complete flat space is homothetic.
(b) M. Obata has communicated to us the following result (unpublished): “A closed infinitesimal conformal transformation of a (locally) reducible Riemannian manifold is homothetic.” This means that only an irreducible Riemannian manifold can admit closed non-homothetic maps.

3. Proof of Theorem 1. It is first shown that

$$\theta(X)\omega + \bar{\theta}(X)\omega = \left(1 - \frac{2}{n}\right)\delta\xi\cdot\omega.$$  

Since $\bar{\theta}(X) = \epsilon(x)\delta + \delta\epsilon(x)$ (where $\epsilon(x)\alpha = \xi \wedge \alpha$ for any $p$-form $\alpha$), $\delta$ and $\theta(X)$ commute. Hence, applying $\delta$ to both sides of (3) we obtain

$$\delta\theta(X)\omega = -\left(1 - \frac{2}{n}\right)C\delta\xi.$$  

Taking the global scalar product with $C\xi$ we derive

$$||\theta(X)\omega||^2 = -\left(1 - \frac{2}{n}\right)||\delta\xi||^2.$$  

For $n > 1$, $\theta(X)\omega$ vanishes. Moreover, $\delta\xi = 0$, that is the automorphisms are isometries.

4. Bochner and Montgomery [1] have shown that the group $G$ of analytic homeomorphisms of a compact complex manifold $M$ is a Lie group. If $M$ is an Einstein Kaehler manifold $G$ is reductive [6]. More generally, if the Ricci scalar curvature of a compact Kaehler manifold is a (positive) constant the same conclusion is valid [5]. This seems to be the best possible generalization of the result of [6] as one may see by considering the Gaussian 2-sphere with any metric with nonconstant scalar curvature. By restricting the analytic maps to those which are closed in the above sense no restrictions of a local nature regarding curvature are required and results parallel to Theorems 3 and 4 may be obtained.

Remark. From the proof of Theorem 3 it follows that the image by $J$ of a closed infinitesimal conformal transformation of a Kaehler manifold is an infinitesimal isometry. In fact
Theorem 5. A closed infinitesimal conformal transformation of a Kaehler manifold is an infinitesimal analytic transformation whose image by $J$ is an infinitesimal isometry.

For noncompact almost Kaehler manifolds we may prove

Theorem 6. If the largest connected group of automorphisms is a semi-simple Lie group its elements are volume preserving transformations.

References


Wayne State University