SPACES OF MEASURABLE TRANSFORMATIONS

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By a space we shall mean a measurable space, i.e. an abstract set together with a \( \sigma \)-ring of subsets, called measurable sets, whose union is the whole space. The structure of a space will be the \( \sigma \)-ring of its measurable subsets. A measurable transformation from one space to another is a mapping such that the inverse image of every measurable set is measurable.

Let \( X \) and \( Y \) be spaces, \( F \) a set of measurable transformations from \( X \) into \( Y \), and \( \phi_F: F \times X \to Y \) the natural mapping defined by \( \phi_F(f, x) = f(x) \). A structure \( R \) on \( F \) will be called admissible if \( \phi_F \), considered as a mapping from the product space \( (F, R) \times X \) into \( Y \), is a measurable transformation.\(^2\) It may not be possible to define an admissible structure on \( F \); if it is, \( F \) itself will also be called admissible. We are concerned with the problem of characterizing, for given \( X \) and \( Y \), the admissible sets \( F \) and the admissible structures \( R \) on the admissible sets.

The following three theorems may be established fairly easily:

**Theorem A.** A set consisting of a single measurable transformation is admissible.

**Theorem B.** A subset of an admissible set is admissible. Indeed, if \( G \subseteq F \), \( R \) is an admissible structure on \( F \), and \( R_G \) is the subspace structure on \( G \) induced\(^3\) by \( R \), then \( R_G \) is admissible on \( G \).

**Theorem C.** The union of denumerably many admissible sets is admissible. Indeed, if \( F = \bigcup_{i=1}^{\infty} F_i \) and \( R_1, R_2, \ldots \) are admissible structures on \( F_1, F_2, \ldots \) respectively, then the structure \( R \) on \( F \) generated by the members of all the \( R_i \) is admissible on \( G \).

Much more can be said if \( X \) and \( Y \) are assumed to be separable, i.e. to have countably generated structures.\(^4\) To state our theorems in this case we first define the concept of Banach class, closely related to that of Baire class. Let \( \mathfrak{A} \) be an arbitrary class of measur-

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\(^1\) The author is much indebted to Professor P. R. Halmos, who suggested a number of significant improvements in the complete version of this note.

\(^2\) \((F, R)\) is the space whose underlying abstract set is \( F \) and whose structure is \( R \).

\(^3\) \( R_G \) consists of all intersections of \( G \) with members of \( R \).

\(^4\) The term is used by analogy with its topological use. We will also use the term "separable structure," meaning a countably generated structure.
urable subsets of \( X \). For each denumerable ordinal number \( \alpha \geq 1 \), we define classes \( P_\alpha(\mathfrak{A}) \) and \( Q_\alpha(\mathfrak{A}) \) inductively as follows: \( Q_1(\mathfrak{A}) \) consists of all denumerable unions of members of \( \mathfrak{A} \), and \( P_1(\mathfrak{A}) \) consists of all complements of members of \( Q_1(\mathfrak{A}) \); supposing \( Q_\beta(\mathfrak{A}) \) and \( P_\beta(\mathfrak{A}) \) to have been defined for all \( \beta < \alpha \), we define \( Q_\alpha(\mathfrak{A}) = Q_1(\bigcup_{\beta < \alpha} P_\beta(\mathfrak{A})) \) and \( P_\alpha(\mathfrak{A}) = P_1(\bigcup_{\beta < \alpha} P_\beta(\mathfrak{A})) \). \( Q_\alpha(\mathfrak{A}) \cup P_\alpha(\mathfrak{A}) \) is the set of all subsets of \( X \) which can be "reached from \( \mathfrak{A} \)" by performing at most \( \alpha \) operations, where each operation consists of forming a denumerable union and a complement. If \( \mathfrak{A} \) generates the structure of \( X \), then the union (over \( \alpha \)) of all the \( Q_\alpha(\mathfrak{A}) \) (or of the \( P_\alpha(\mathfrak{A}) \)) is the set of all measurable subsets of \( X \). If \( \mathfrak{B} \) is a class of measurable subsets of \( Y \) and \( \alpha \geq 0 \) is a denumerable ordinal number, then we define \( L_\alpha(\mathfrak{A}, \mathfrak{B}) \) to be the set of all functions \( f: X \to Y \) such that for all \( \mathfrak{B} \subseteq Q_1(\mathfrak{A}) \), \( f^{-1}(\mathfrak{B}) \subseteq Q_{\alpha+1}(\mathfrak{A}) \). If \( X \) and \( Y \) are separable and \( \mathfrak{A} \) and \( \mathfrak{B} \) are denumerable generating sets for their respective structures, then the union (over \( \alpha \)) of all the \( L_\alpha(\mathfrak{A}, \mathfrak{B}) \) is the set of all measurable transformations from \( X \) into \( Y \). It will be denoted \( Y^X \). In this case \( L_\alpha(\mathfrak{A}, \mathfrak{B}) \) is called the Banach class\(^6\) of order \( \alpha \) for \((\mathfrak{A}, \mathfrak{B})\). A subset \( F \) of \( Y^X \) is said to be of bounded Banach class if there is an \( \alpha \) and denumerable generating sets \( \mathfrak{A}, \mathfrak{B} \) such that \( F \subseteq L_\alpha(\mathfrak{A}, \mathfrak{B}) \). It is important to note that the definition of bounded Banach class is independent of the choice of \( \mathfrak{A} \) and \( \mathfrak{B} \), i.e. that if \( F \subseteq L_\alpha(\mathfrak{A}, \mathfrak{B}) \), then for any other generating pair \( \mathfrak{A}', \mathfrak{B}' \), there is an \( \alpha' \) such that \( F \subseteq L_{\alpha'}(\mathfrak{A}', \mathfrak{B}') \). If \( X \) and \( Y \) are separable metric spaces and \( Y \) is pathwise connected, then the Banach classes coincide with the Baire classes (for appropriate choice of \( \mathfrak{A} \) and \( \mathfrak{B} \)).

**Theorem D.** If \( X \) and \( Y \) are separable, then \( F \) is admissible if and only if it is of bounded Banach class.

**Theorem E.** If \( X \) and \( Y \) are separable, then every admissible subset of \( Y^X \) has a separable admissible structure.

A space \( Z \) and its structure are called regular if for all \( x, y \in Z \), there is a measurable set in \( Z \) containing \( x \) but not \( y \). It is known (cf. \([2]\)) that a space is separable and regular if and only if it is isomorphic\(^6\) to a subspace of \( I \), where \( I \) denotes the unit interval \([0, 1]\) with the usual Borel structure.

**Theorem F.** If \( X \) and \( Y \) are separable and regular, then every admissible subset of \( Y^X \) has a separable and regular admissible structure.

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\(^6\) Because of the work that Banach \([1]\) did in characterizing these classes.

\(^6\) Two spaces are said to be isomorphic if there is a \( 1-1 \) correspondence between them that preserves measurability (in both directions).
The natural admissible structure on a given admissible set $F$ is defined to be the smallest admissible structure on $F$, if it exists. Alternatively, it may be defined to be the intersection of all the admissible structures on $F$, in case this is admissible. Not every admissible set need have a natural admissible structure; the counterexample is due to P. R. Halmos.

If $a \in X$ and $B \subset Y$, define $F(a, B) = \{ f : f \in F, f(a) \in B \}$. It is not hard to prove that if $B$ is measurable and $a$ is arbitrary, then every admissible structure on $F$ must contain $F(a, B)$. A "converse" would be that the structure generated by the $F(a, B)$ is admissible, and it would follow that it is also natural.

**Theorem G.** If $X$ and $Y$ are separable metric spaces and $F$ contains continuous functions only, then $F$ has a natural admissible structure, which is generated by the set of all $F(a, B)$, where $B$ is measurable and $a$ is arbitrary.

We now give some applications. A space is said to have the discrete structure if every subset is measurable. Let $J$ be the space consisting of 0 and 1 only, and $K$ the space of all positive integers, both with the discrete structure. If $X$ is an arbitrary space, then $X^I$ and $X^K$ are both admissible, and possess natural admissible structures which make them isomorphic to $XXX$ and $XXX$ respectively, where the $X_i$ are copies of $X$. In particular, $J^K$ is admissible and has a natural admissible structure which makes it isomorphic to $I$. These results are relatively trivial or at least easily derivable from known results.

The situation changes when we pass to exponent spaces with non-discrete structures. For example, $J^I$ may be considered the set of all measurable subsets of $I$. It is not itself admissible. The set of all open subsets of $I$ is admissible, as is the set of all closed subsets, the set of all $G_\delta$, etc. In general, a subset $F$ of $J^I$ is admissible if and only if all members of $F$ can be constructed from the open subsets of $I$ by taking denumerable unions and intersections at most $\alpha$ times, where $\alpha$ is an arbitrary denumerable ordinal number (which is fixed for given $F$, but may differ for different $F$). I do not know whether or not every admissible subset of $J^I$ has a natural admissible structure, but if $F$ is admissible, then we may endow it with an admissible structure in such a way so that it will be isomorphic to a subset of $I$.

$I^I$ is not admissible. The set of all continuous functions from $I$ into $I$ is admissible; more generally, a necessary and sufficient condition that a subset $F$ of $I^I$ be admissible is that there exist a denumerable ordinal number $\alpha$ such that all members of $F$ are of Baire class $\alpha$. 

at most. The set $H$ of all continuous functions from $I$ into $I$ has a natural admissible structure; it is the Borel structure of $H$ when considered as a metric space (in the uniform convergence topology). Again, I do not know whether or not every admissible subset of $I^I$ has a natural admissible structure, but if $F$ is admissible, we may endow it with an admissible structure in such a way so that it will be isomorphic to a subset of $I$.

The above theory may be applied to give a generalization of Kuhn's theorem [3] about optimal behavior strategies in games of perfect recall, to games in which there may be a continuum of alternatives at some of the moves.

A fuller account of the theory outlined above, together with proofs, will be published elsewhere.

References


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