LOOP SPACES OF $H$-SPACES$^1$

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Let $Y$ be an $H$-space (a space $Y$ with a continuous product $Y \times Y \rightarrow Y$ which has a unit element), $Y$ arcwise connected and simply connected, and let $X = \Omega Y$, the space of loops of $Y$ based at the unit. We will prove

**Theorem 1.** If $H^\ast(X)$ (the singular cohomology ring over the integers) is a finitely generated module over the integers, then $X$ is of the same singular homotopy type as $K(G, 1)$ where $G$ is a free abelian group ($K(G, 1) = S^1 \times \cdots \times S^1 =$ the $n$-torus, where rank of $G = n$).

Thus the loop space of an $H$-space $Y$ is infinite dimensional unless $Y = K(G, 2)$, $G$ free abelian.

The proof depends on Theorem 2 below.

Let $p$ be a prime. Then the cohomology of an $H$-space $Y$ over $\mathbb{Z}_p$ is a Hopf algebra (see [2]). If $\psi: Y \times Y \rightarrow Y$ is the multiplication in $Y$, then $\psi: H^\ast(Y; \mathbb{Z}_p) \rightarrow H^\ast(Y; \mathbb{Z}_p) \otimes H^\ast(Y; \mathbb{Z}_p)$ is the diagonal map of the Hopf algebra, the product being the cup product.

Let $A$ be a Hopf algebra over $\mathbb{Z}_p$, $\psi: A \rightarrow A \otimes A$ the diagonal map, $\theta: A \otimes A \rightarrow A$ the product. An element $x \in A$ is called primitive if $\psi(x) = x \otimes 1 + 1 \otimes x$. An element $y \in A$ is called decomposable if $y \in \theta(\mathcal{A} \otimes \mathcal{A})$ where $\mathcal{A}$ is the subspace of $A$ consisting of positive dimensional elements. Let $P(A)$ denote the primitive elements of $A$, $D(A)$ the decomposable elements of $A$, $Q(A) = A / D(A)$. Let $\xi: A \rightarrow A$ be defined by $\xi(x) = x^p$. Then $\xi(A)$ is a Hopf subalgebra of $A$.

We quote a theorem of Milnor and Moore [9].

**Theorem (Milnor and Moore).** Let $A$ be an associative, commutative Hopf algebra over $\mathbb{Z}_p$ with $A_0 = \mathbb{Z}_p$. Then the sequence $0 \rightarrow P(\xi A) \rightarrow \xi P(A) \rightarrow \xi Q(A)$ is exact.

Thus if $x \in P(A) \cap D(A)$, then $x = u^n$ for some $u \in A$.

Let $\sigma^i$ denote the $i$th Steenrod operation

$$\sigma^i: H^n(X; \mathbb{Z}_p) \rightarrow H^{n+2i}(X; \mathbb{Z}_p) \quad (p \text{ an odd prime}),$$

$Sq^i$ denote the $i$th Steenrod square

$$Sq^i: H^n(X; \mathbb{Z}_2) \rightarrow H^{n+i}(X; \mathbb{Z}_2).$$

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THEOREM 2. Let $Y$ be an $H$-space with $H^*(Y; \mathbb{Z}_p) = \mathbb{P}(y_1, \ldots , y_n, \ldots )$ be the ring of polynomials over $\mathbb{Z}_p$ generated by $y_1, \ldots , y_n, \ldots$, with $\dim y_i$ even for all $i$. Let $x \in H^{2m}(Y; \mathbb{Z}_p)$ be a primitive element. If $p \neq 2$ and $m = p^r + k$ with $0 < k < p$, and $r > 0$, then $\varphi^{p^r+i}(x) \neq 0$ is indecomposable, $0 \leq i < k$. If $p \neq 2$ and $1 < m < p$ $\varphi^i(x) \neq 0$ is indecomposable, $0 < i < m$. If $p = 2$, and $m = 2^{r-1} + 1$ with $r > 1$, then $Sq^{2r}(x)$ is indecomposable.

PROOF. Since $H^*(Y; \mathbb{Z}_p)$ is a polynomial ring $\mathbb{P}^m(x) = x^p \neq 0$. Let $p \neq 2, m = p^r + k$, $k < p$, $r > 1$. Then by the Adem relations [7] $\varphi^{p^r+i} = i!(\varphi^1)^{k} \varphi^{p^r}$ so that $\varphi^{p^r+i}(x) = i!(\varphi^1)^{i} \varphi^{p^r}(x) \neq 0$ for $0 \leq i \leq k < p$. Now if $\alpha$ is an element of the Steenrod algebra and $Y$ is an $H$-space, then $\alpha(P(H^*(Y; \mathbb{Z}_p))) \subseteq P(H^*(Y; \mathbb{Z}_p))$ since $\alpha$ is additive, so that $\varphi^{p^r+i}(x)$ is primitive in $H^*(Y; \mathbb{Z}_p)$. If $\varphi^{p^r+i}(x)$ is decomposable then by the theorem of Milnor and Moore $\varphi^{p^r+i}(x) = u^p$. But $\varphi^1(\varphi^{p^r+i}) = (i+1)\varphi^{p^r+i+1}$ so that if $i < k$ $\varphi^1(u^p) \neq 0$. But $\varphi^1$ is a derivation by the Product Formula so that $\varphi^1(u^p) = pu^{p-1}(\varphi^1 u) = 0 \mod p$. Hence $\varphi^{p^r+i}(x)$ is indecomposable in $H^*(Y; \mathbb{Z}_p)$.

If $1 < m < p$, $p \neq 2$, we get from the Adem relations $\varphi^m = m! (\varphi^1)^m$, and we proceed similarly.

If $p = 2$ and $m = 2^{r-1} + 1$ with $r > 1$ we have that $x^2 = Sq^{2r+2}(x)$ and from the Adem relations

$$ Sq^{2r+2} = Sq^2Sq^{2r} + Sq^{2r+1}Sq^1. $$

Since $H^*(Y; \mathbb{Z}_2)$ is a polynomial ring on even dimensional generators $Sq^1H^*(Y; \mathbb{Z}_2) = 0$, for $Sq^1$ changes dimension by 1. Hence $Sq^{2r+2}(x) = Sq^2Sq^{2r}(x)$ so that $Sq^2(x) \neq 0$ and is a primitive element. If $Sq^2(x) = u^2$ then $Sq^2(u^2) = (Sq^2u)u + (Sq^1u)(Sq^1u) + u(Sq^2u) = 0 \mod 2$, since $Sq^1 = 0$ in $H^*(Y; \mathbb{Z}_2)$ ($Sq^2$ is a derivation on $H^*(Y; \mathbb{Z}_2)$). Hence $Sq^2(x)$ is indecomposable. Q.E.D.

One can apply Theorem 2 to compute many Steenrod operations in the stable classical groups, using only the cohomology structure mod $p$ and the fact that the classifying space is an $H$-space. We will use Theorem 2 to prove Theorem 1.

PROOF OF THEOREM 1. Let $\overline{X}$ = the universal covering space of $X$. Then it follows from the results of [3] that $H^*(\overline{X})$ is finitely generated. Further, $\overline{X} = \Omega \overline{Y}$ where $\overline{Y}$ is the 2-connected fibre space over $Y$ (see [10]). Further $\overline{Y}$ is the fibre of a multiplicative fibre map of $Y$ into $K(\pi_2(Y), 2)$, and hence $\overline{Y}$ is an $H$-space. We will show that $\overline{X}$ is acyclic, (i.e., that $H^i(\overline{X}) = 0$ for $i > 0$) and therefore $X$ is a $K(\pi, 1)$,
finite dimensional with $\pi$ abelian finitely generated. Then $\pi$ must be free abelian and the result will be achieved.

Therefore we will assume that $\pi_1(X)=0$ and show that $X$ is acyclic.

If $X$ is not acyclic, then $H^*(X)/\text{Torsion}$ is nontrivial (see Part 1 of Theorem 3 of [4] or see [5]), and hence $H^*(X)/\text{Torsion} = \Lambda(x_1, \cdots, x_n)$, the exterior algebra on odd dimensional generators $x_1, \cdots, x_n$ (see [2]). Since $H^*(X)$ is finitely generated, only a finite number of primes occur as torsion numbers of $H^*(X)$. Hence for almost all primes, in particular for all sufficiently large primes $p$, $H^*(X; \mathbb{Z}_p) = (H^*(X)/\text{Torsion}) \otimes \mathbb{Z}_p$. Therefore we have $H^*(X; \mathbb{Z}_p) = \Lambda(x_1, \cdots, x_n)$ (identifying $x_i$ with its image in $(H^*(X)/\text{Torsion}) \otimes \mathbb{Z}_p = H^*(X; \mathbb{Z}_p)$) for all sufficiently large $p$.

By a theorem of Borel (Theorem 13.1 of [2]) we have that $H^*(Y; \mathbb{Z}_p) = P(y_1, \cdots, y_n)$ if the prime $p$ is not a torsion number of $H(X)$, with $\dim y_i = \dim x_i + 1$. Let the $y$'s be ordered so that $2k = \dim y_1 \leq \dim y_i \leq \dim y_n = 2m$, $1 \leq i \leq n$, and $k > 1$ since $\dim x_i > 2$ for all $i$.

Choose $p$ so large that $2k + 2(p - 1) > 2m$ and $p > k > 1$, or in other words choose $p > \max(m - k - 1, k)$, and large enough that $p$ does not occur as a torsion number of $H^*(X)$. Then $y_1$ is primitive since it is in the first nonvanishing cohomology group of $Y$ and we may apply Theorem 2 to $y_1 \in H^*(Y; \mathbb{Z}_p)$. Hence $\varphi^1(y_1) \neq 0$ and is an indecomposable element in $H^*(Y; \mathbb{Z}_p)$. But $\dim \varphi^1(y_1) = 2k + 2(p - 1) > 2m$, and all elements of $H^q(Y; \mathbb{Z}_p)$ are decomposable if $q > 2m$. This is a contradiction, so $X$ is acyclic. Q.E.D.

One might conjecture that if $X$ is a homotopy commutative $H$-space and $H^*(X)$ is finitely generated then $X$ is of the same singular homotopy type as $K(G, 1)$ with $G$ a free abelian group. Araki, James and Thomas have shown that the usual multiplication on a compact Lie group $G$ is not homotopy commutative unless $G$ is a torus [1], and James [8] has shown that the spheres $S^3$ and $S^7$ have no homotopy commutative multiplications. It will be shown elsewhere [6] that if $X$ is a homotopy commutative $H$-space with $H^*(X)$ finitely generated, then $H^*(X)$ has no 2-torsion. Hence the Lie groups which have 2-torsion (such as $SO(n)$ and the exceptional groups) have no homotopy commutative multiplications on them. It will also be shown in [6] that if $X$ is homotopy associative and homotopy commutative, and $H^*(X)$ is finitely generated, then $H^*(X)$ has no torsion, so that $H^*(X) = \Lambda(x_1, \cdots, x_n)$.

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BIBLIOGRAPHY


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