AN EXACT SEQUENCE IN DIFFERENTIAL TOPOLOGY

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1. Introduction. The purpose of this note is to describe an exact sequence relating three series of abelian groups: $\Gamma^n$, defined by Thom [3]; $\theta^n$, defined by Milnor [1]; and $\Lambda^n$, defined below. The sequence is written

$\cdots \to \Gamma^n \xrightarrow{j} \theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \to \cdots$ (1)

We now describe these groups briefly.

To obtain $\Gamma^n$, divide the group of diffeomorphisms of the $n-1$ sphere $S^{n-1}$ by the normal subgroup of those diffeomorphisms that are extendable to the $n$-ball. See [2] for details.

The set $\theta^n$ is the set of $J$-equivalence classes of closed, oriented, differentiable $n$-manifolds that are homotopy spheres. If $M$ is an oriented manifold, let $-M$ be the oppositely oriented manifold. Two closed oriented $n$-manifolds $M$ and $N$ are $J$-equivalent if there is an oriented $n+1$-manifold $X$ whose boundary is the disjoint union of $M$ and $-N$, and which admits both $M$ and $N$ as deformation retracts. We denote the $J$-equivalence class of $M$ by $[M]$. If $[M]$ and $[N]$ are elements of $\theta^n$, their sum is defined to be $[M \# N]$, where $[M \# N]$ is obtained by removing the interior of an $n$-ball from $M$ and $N$ and identifying the boundaries in a suitable way. Details may be found in [1].

The group $\Lambda^n$ is defined analogously using combinatorial manifolds. Instead of the interior of an $n$-ball, the interior of an $n$-simplex is removed. If $M$ is a combinatorial manifold, we write $\langle M \rangle$ for its $J$-equivalence class.

2. The sequence. To define $k: \theta^n \to \Lambda^n$, we observe that every differentiable manifold $M$ defines a combinatorial manifold $\overline{M}$, unique up to combinatorial equivalence, by means of a smooth triangulation of $M$ [4]. We define $k[M] = \langle \overline{M} \rangle$.

Let $g: S^{n-1} \to S^{n-1}$ represent an element $\gamma$ of $\Gamma^n$. According to J. Munkres [2], there is a unique (up to diffeomorphism) differentiable manifold $V_\gamma$ corresponding to $\gamma$, such that $\overline{V_\gamma} = S^n$. To obtain $V_\gamma$, identify two copies of $R^n - \{0\}$ by the diffeomorphism $x \mapsto (1/|x|)g(x/|x|)$. Here $R^n$ is Euclidean $n$-space and $|x|$ is the usual norm. The diffeomorphism class of $V_\gamma$ depends only on $\gamma$, and
$V_\gamma$ is diffeomorphic to $V_\delta$ if and only if $\gamma = \delta$. We define $j: \Gamma^n \to \Theta^n$ by $j(\gamma) = [V_\gamma]$.

To define $d: \Lambda^n \to \Gamma^{n-1}$, we proceed as follows. If $\langle M \rangle$ is an element of $\Lambda^n$, let $M_0$ be obtained from $M$ by removing the interior of an $n$-simplex. According to a recent result of A. M. Gleason, there is a differentiable manifold $N$ such that $\overline{N} = M_0$. By [2], $N$ is unique up to diffeomorphism, because $M_0$ is contractible. Since the boundary $\partial N$ of $N$ is combinatorially an $n-1$ sphere, there is a unique $\beta \in \Gamma^{n-1}$ such that $\partial N = V_\beta$. We define $d(\langle M \rangle) = \beta$. It can be shown that $\beta$ depends only on $\langle M \rangle$.

**Theorem.** The sequence (1) is exact.
The proof will appear in a subsequent paper.

**Bibliography**

1. J. Milnor, *Differentiable manifolds which are homotopy spheres*, Princeton University, 1959 (mimeographed).

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