HOMOTOPY-ABELIAN LIE GROUPS

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A topological group \( G \) is said to be \textit{homotopy-abelian} if the commutator map of \( G \times G \) into \( G \) is nulhomotopic. Examples can be given\(^2\) of non-compact Lie groups which are homotopy-abelian but not abelian. The purpose of this note is to prove

\textbf{Theorem.} A compact connected Lie group is homotopy-abelian only if it is abelian.

\textbf{Corollary.} If a Lie group is homotopy-abelian, then its maximal compact connected subgroup is abelian.

Our proof depends on the theory of \([\text{?}]\). Thus we consider the Samelson “commutator” product\(^3\) in the homotopy groups of \( G \), which is trivial when \( G \) is homotopy-abelian. The product of \( \alpha \in \pi_p(G) \) with \( \beta \in \pi_q(G) \) is denoted by \( \langle \alpha, \beta \rangle \in \pi_{p+q}(G) \), where \( p, q \geq 1 \). If \( h \) is a homomorphism of \( G \) into another topological group then

\[ h_\# \langle \alpha, \beta \rangle = \langle h_\# \alpha, h_\# \beta \rangle, \]

where \( h_\# \) denotes the induced homomorphism. Note that \( h_\# \) is an isomorphism if \( h \) is a covering map and \( p, q \geq 2 \). Hence if two topological groups have a common universal covering group then their higher homotopy groups are related by an isomorphism which is compatible with the Samelson product. Let \( \sigma \pi_q(G) \), where \( q \geq 1 \), denote the subset of \( \pi_{2q}(G) \) consisting of elements \( \langle \beta, \beta \rangle \), where \( \beta \in \pi_q(G) \). We assert the following

\textbf{Lemma.} Let \( G \) be a compact connected simple non-abelian Lie group of dimension \( n \) and rank \( l \). Then \( \sigma \pi_q(G) \neq 0 \), where \( q = 2n/l - 3 \).

The proof is by application of (2.2) of \([\text{6}]\). We distinguish between the classical and exceptional cases, beginning with the latter.

Let \( G \) be one of the exceptional groups. Then \( n/l = p \), an odd prime number, and \( G \) has no \( p \)-torsion (see \([\text{3}]\)). The mod \( p \) cohomology of \( G \) is an exterior algebra on a basis of \( l \) generators. There is one generator \( y \) in dimension \( q \), while the remainder are of lower dimension. It follows from Proposition 6 on page 291 of \([\text{8}]\) that \( y \) has a nontrivial

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\(^2\) Such as the 2-dimensional affine group (example suggested by H. Samelson).
\(^3\) The theory of the Samelson product is given in \([\text{5}]\), for example.
image under the homomorphism induced by some map of $S^q$ into $G$. Thus $y$ has nonzero index, in the sense of $[6]$, with regard to some element $\beta \in \pi_q(G)$. By Borel's theorem the mod $p$ cohomology of $B$, the classifying space of $G$, is a polynomial algebra on a basis of $l$ generators which correspond under transgression to those of the exterior algebra. The generator $x$ corresponding to $y$ has a nontrivial image under the homomorphism induced by some map of $S^{q+1}$ into $B$. In the polynomial algebra let $M$ denote the ideal generated by all the basis elements except $x$. If $z$ is such a generator then

$$\dim z < \dim x = q + 1 = 2(p - 1),$$

and so $\vartheta^s z \in M$, where $\vartheta^s (s \geq 0)$ denotes the Steenrod operator. Hence $\vartheta^p M \subset M$, by the Cartan product formula. This proves that $\vartheta^1 x \notin M$, since by the Adem relation $[1]$ we have

$$\vartheta^{p-2} \vartheta^1 x = (p - 1)\vartheta^{p-1} x = (p - 1)x^p \in M.$$

Hence $\vartheta^1 x = cx^2$, mod $M$, where $c \neq 0$, and so $\vartheta^1 x$ is significant with regard to $\beta$, in the sense of $[6]$. Therefore $\langle \beta, \beta \rangle \neq 0$, by (2.2) of $[6]$, which proves the lemma when $G$ is exceptional.

If $G$ is not exceptional then $G$ is locally-isomorphic to one of the classical groups:

$$SU(l + 1), \quad SO(2l + 1), \quad Sp(l), \quad SO(2l).$$

It is shown in §4 of $[6]$ that each of

$$\sigma\pi_{2l+1}U(l + 1), \quad \sigma\pi_{4l-1}SO(2l + 1), \quad \sigma\pi_{4l-1}Sp(l),$$

contains elements of odd order, and it follows from (18.2) of $[4]$ that the same is true of $\sigma\pi_{4l-5}SO(2l) (l \neq 1)$. Furthermore

$$\pi_r SU(l + 1) \approx \pi_r U(l + 1), \quad (r \geq 2),$$

under the injection, and so $\sigma\pi_{2l+1}SU(l+1) \neq 0$. Since the Samelson product is an invariant of the structure class this completes the proof of the lemma.

To deduce the theorem we recall that a compact connected Lie group $G$ is locally isomorphic to $G'$, say, where $G'$ is the direct product of an abelian group $T$ with various nonabelian simple groups. When any of these latter are present there exists, by the lemma, some value of $q$ such that $\sigma\pi_q(G') \neq 0$ and hence $\sigma\pi_q(G) \neq 0$. Thus $G' = T$ if $G$ is homotopy-abelian, and hence the theorem follows at once. A maximal

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4 See (7.2) and (19.1) of $[2]$. 
compact connected subgroup of a Lie group is a deformation retract of the component of the identity [7], and so the corollary is an immediate consequence of the theorem.

REFERENCES

5. I. M. James, On H-spaces and their homotopy groups, (to be published in Oxford Quart. J. of Math.).