RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A POLYNOMIAL CANONICAL FORM FOR CERTAIN ANALYTIC FUNCTIONS OF TWO VARIABLES AT A CRITICAL POINT

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THEOREM. Let \( F(z, w) \) be analytic for small \( |z| \) and \( |w| \) and \( F(0, 0) = 0 \). Then (Weierstrass Preparation Theorem)

\[
F(z, w) = z^k[w^m + a_1(z)w^{m-1} + \cdots + a_m(z)]\Phi(z, w)
\]

where \( \Phi(0, 0) \neq 0 \) and \( a_j(0) = 0 \). Let the discriminant of the polynomial in \( w \), in the bracket above, not vanish identically (so that there are no quadratic factors of \( F \) which are polynomials in \( w \)). Then there exists \( \psi(\xi, \omega) \) a polynomial in \( (\xi, \omega) \) of degree \( m \) in \( \omega \) and analytic functions \( \gamma(z, w) \) and \( \delta(z, w) \) such that \( \gamma(0, 0) = \partial \gamma / \partial z(0, 0) = \partial \gamma / \partial w(0, 0) = 0 \) and similarly for \( \delta \) such that if

\[
\xi = z + \gamma(z, w), \quad \omega = w + \delta(z, w)
\]

then

\[
\psi(\xi, \omega) = F(z, w).
\]

(Note that \( \psi \) is a polynomial in both variables.) An outline of the proof follows.

By [1] it is known that \( F \) can be transformed by use of (2) to the form of (1) with \( \Phi = 1 \). Hence the case

\[
F(z, w) = f_0(z)w^m + f_1(z)w^{m-1} + \cdots + f_m(z)
\]

where \( f_0 = z^k \) and \( z^{k+1} | f_j(z) \) \( j \geq 1 \), can be considered.

Because of the hypothesis on \( F \) it can be shown that the resultant of \( F_z = \partial F / \partial z \) and \( F_w \) does not vanish identically. Thus

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\( D(z) = \begin{bmatrix}
0 & 0 & \cdots & 0 & f'_0 & 0 & \cdots & 0 & mf_0 \\
0 & 0 & \cdots & f'_0 & f'_1 & 0 & \cdots & mf_0 (m - 1)f_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f'_{m-1} & f'_{m} & \cdots & 0 & 2f_{m-2} & \cdots & 0 & 0 \\
f'_{m} & 0 & \cdots & 0 & f_{m-1} & \cdots & 0 & 0
\end{bmatrix} \neq 0 \)

for small \(|z| > 0\). Let the lowest nonvanishing power of \(z\) in \(D(z)\) be \(z^k\). Let \(P_0(z) = z^k\) and for \(j \geq 1\) let \(P_j(z)\) be polynomials of degree \(2\mu + 2\) such that \(P_j - f_j\) has least power of \(z\) of degree at least \(2\mu + 3\). Let the polynomial

\[ \psi(z, \omega) = z^k \omega^m + P_1(z) \omega^{m-1} + \cdots + P_m(z). \]

Consider now the equation

\[ \psi(z + g, w + h) = \psi(z, w) + g\psi_z(z, w) + h\psi_w(z, w) + R(z, w, g, h) \]

where each term in the polynomial \(R\) is of degree at least two in \((g, h)\). Hence (8) can be written as

\[ (g_0 + wg_1 + \cdots + w^{m-2}g_{m-2})\psi_z(z, w) \]

\[ + (h_0 + \cdots + w^{m-1}h_{m-1})\psi_w(z, w) \]

\[ = (f_1 - P_1)w^{m-1} + \cdots + (f_m - P_m) \]

\[ - R(z, w, g_0 + \cdots + g_{m-2}w^{m-2}, h_0 + \cdots). \]

Certainly the equation (7) will be satisfied if the coefficients of \(w^i\) on the left are set equal to those of \(w^i\) on the right except that \(-R\) is kept with \(f_m - P_m\) so that the \(2m - 1\) equations obtained from (7) are

\[ P'_0(z)g_{m-2} + mP_0(z)h_{m-1} = 0, \cdots, \]

\[ P'_m g_0 + P_{m-1}h_0 = f_m - P_m - R. \]

Because of (5) and the coincidence of the early terms of \(P_j\) and \(f_j\), the equations (8) can be inverted to give

\[ g_i = z^{-\nu} \sum_{j=1}^{m} \alpha_{ij}(z)(f_j - P_j) - z^{-\nu} \alpha_{im} R, \quad i = 0, \cdots, m - 2 \]

\[ h_i = z^{-\nu} \sum_{j=1}^{m} \beta_{ij}(z)(f_j - P_j) - z^{-\nu} \beta_{im} R, \quad i = 0, \cdots, m - 1 \]
where $\alpha_{ij}$ and $\beta_{ij}$ are analytic in $z$. Next let $g_i = z^{p_i+1}u_i$ and $h_i = z^{q_i+1}v_i$. If

$$R(z, w, z^{p_0+1}u_0 + \cdots + z^{p_{m-2}+1}w^{m-2}, z^{p_{m-1}+1}v_0 + \cdots + z^{p_{m-1}+1}w^{m-1})$$

then $\bar{R}$ is a polynomial in all variables of degree at least two in $(u_i, v_i)$. Hence (9) and (10) become

$$u_i + z\alpha_{im}(z)\bar{R} = \sum_{i=1}^{m} \alpha_{ij}(z)z^{-2^{i-1}}(f_i - P_i), \quad i = 0, \cdots, m - 2,$$

(11)

$$v_i + z\beta_{im}\bar{R} = \sum_{i=1}^{m} \beta_{ij}(z)z^{-2^{i-1}}(f_i - P_i), \quad i = 0, \cdots, m - 1.$$  

Since $(f_i - P_i)z^{-2^{i-1}}$ is analytic and vanishes at $z=0$, and since $u_i = v_i = 0$ is a solution of (11) for $z = w = 0$, it follows from the implicit function theorem that for small $|z|$ and $|w|$, (11) has an analytic solution $u_i(z, w), v_i(z, w)$.

The question of whether it was possible to extend the result of [1] to the form of a polynomial $\psi$ in both variables (rather than in just one as in [1]) was asked of me by Felix Browder.

**Reference**


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