CONFORMAL TRANSFORMATIONS IN RIEMANNIAN AND HERMITIAN SPACES

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The purpose of the present note is to show that the results recently announced by S. I. Goldberg [1] in this Bulletin are valid also in slightly more general forms.

1. Consider a conformal Killing vector \( v^h \) in an \( n \)-dimensional Riemannian space. Then the Lie derivative of the fundamental tensor \( g_{ij} \) and that of Christoffel symbols with respect to \( v^h \) are respectively given by

\[
\mathcal{L}_v g_{ij} = \nabla_j v_i + \nabla_i v_j = 2\phi g_{ij},
\]

and

\[
\mathcal{L}_v \left\{ v^j \right\}_{j} = \nabla_j v^h + K_{hij} v^k = A^h v_i + A^h v_j - \phi g_{ij},
\]

where \( \nabla_j \) is the symbol of covariant differentiation, \( K_{hij} \) the curvature tensor, \( A^h \) the unit tensor and \( \phi = \nabla_i \phi, \phi^h \) being its contravariant components.

For a skew-symmetric tensor \( w_{i_1 i_2 \ldots i_l} \), we have in general [5]

\[
\mathcal{L}_v \nabla_j w_{i_1 i_2 \ldots i_l} - \nabla_j \mathcal{L}_v w_{i_1 i_2 \ldots i_l} = -\left( \mathcal{L}_v \left\{ \begin{array}{c} j \\ i_l \end{array} \right\} \right) w_{i_1 i_2 \ldots i_{l-1}} - \cdots - \left( \mathcal{L}_v \left\{ \begin{array}{c} j \\ i_1 \end{array} \right\} \right) w_{i_2 i_3 \ldots i_l}.
\]

Taking the skew-symmetric part with respect to \( j, i_1 \ldots i_l \), we find

\[
\mathcal{L}_v \nabla_{[j} w_{i_1 i_2 \ldots i_l]} = \nabla_{[j} \mathcal{L}_v w_{i_1 i_2 \ldots i_l]},
\]

from which

**Theorem 1.1.** The Lie derivative of a closed skew-symmetric tensor is closed.

Transvecting (1.3) with \( g^{i_1 i_2} \) and taking account of (1.1) and (1.2), we get

\[
\mathcal{L}_v g^{i_1 i_2} \nabla_j w_{i_1 i_2 \ldots i_l} + 2\phi g^{i_1 i_2} \nabla_j w_{i_1 i_2 \ldots i_l} - g^{i_1 i_2} \nabla_j \mathcal{L}_v w_{i_1 i_2 \ldots i_l} \]

\[
= (n - 2p) \phi^i w_{i_1 i_2 \ldots i_l},
\]

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from which

**Theorem 1.2.** The Lie derivative of a coclosed skew-symmetric tensor of order $p$ with respect to a conformal Killing vector is coclosed if and only if $p = n/2$, $n$ being even, or $\nabla^i(\phi w_{i_{p-1} \ldots i_1}) = 0$, that is, $\phi w_{i_{p-1} \ldots i_1}$ is also coclosed, where $\phi$ is the function appearing in $\mathcal{R} g_{ij} = 2\phi g_{ij}$.

Combining Theorems 1.1 and 1.2 we have

**Theorem 1.3.** The Lie derivative of a harmonic tensor $w$ of order $p$ in an $n$-dimensional Riemannian space with respect to a conformal Killing vector is also harmonic if and only if $p = n/2$, $n$ being even, or $\phi w$ is coclosed.

The most specific statement resulting is as follows, see [4; 5; 6].

**Theorem 1.4.** The Lie derivative of a harmonic tensor $w$ of order $p$ in an $n$-dimensional compact orientable Riemannian space with respect to a conformal Killing vector is zero if and only if $p = n/2$, $n$ being even, or $\phi w$ is coclosed where $\phi$ is a function appearing in $\mathcal{R} g_{ij} = 2\phi g_{ij}$ [1].

2. In an almost complex space, a contravariant almost analytic vector is defined as a vector $\nu^h$ which satisfies

$$\mathcal{L}_v F_i^h = v^t \partial_t F_i^h - F_i^t \partial_t \nu^h + F_i^h \partial_t \nu^t = 0. \tag{2.1}$$

In an almost Hermitian space, (2.1) may be written as

$$\mathcal{L}_v F_i^h = v^t \nabla_t F_i^h - F_i^t \nabla_t \nu^h + F_i^h \nabla_t \nu^t = 0, \tag{2.2}$$

from which, by a straightforward calculation,

$$\nabla^i \nabla_t \nu^h + K_i^h \nu^t - F_i^h (\mathcal{R} F_i^t) - \frac{1}{2} F_{j;h} (\mathcal{R} F_j^t) = 0, \tag{2.3}$$

where $K_i^h$ is the Ricci tensor and

$$F_i^t = \nabla^j F_j^i,$$

$$F_{j;h} = \nabla_j F_{i;h} + \nabla_i F_{j;h} + \nabla_h F_{ji}.$$ 

If we put

$$S^i_t = g^j_t (\mathcal{R} F_t^i),$$

and suppose that the space is compact, we have
\[ \int \left[ \left\{ \nabla^i \nabla^h + K^h_{\ i} v^i - F^h_i (\mathcal{Q} s^i) - \frac{1}{2} F^h_{ji} (\mathcal{Q}_s s^i) \right\} v^h + \frac{1}{2} S^{ij} S^i_{\ ji} \right] d\sigma = 0, \]  

(2.4)
d\sigma being volume element of the space.

From (2.3) and (2.4) we have

**Theorem 2.1.** A necessary and sufficient condition for a vector \( v^h \) in a compact almost Hermitian space to be contravariant analytic is (2.3).

Suppose that a conformal Killing vector \( v^h \) satisfies

\[ ^W + F^h_{ji} (\mathcal{Q} s^i) = 0. \]

Substituting

\[ \nabla^i \nabla^h + K^h_{\ i} v^i = - \frac{n-2}{n} \nabla^h (\nabla^h) \]

obtained from (1.2) into (2.4), we find

\[ \int \left[ \frac{n-2}{n} (\nabla^h)^2 + \frac{1}{2} S^{ij} S^i_{\ ji} \right] d\sigma = 0, \]

(2.5)

from which, for \( n>2 \),

\[ \nabla^h v^i = 0, \quad S^i_{\ ji} = 0 \]

and consequently \( v^h \) is a Killing vector \([4;6]\) and at the same time a contravariant almost analytic vector, and for \( n=2 \), we have \( S^i_{\ ji} = 0 \). Thus we have

**Theorem 2.2.** If a conformal Killing vector \( v^h \) in an \( n \)-dimensional compact almost Hermitian space satisfies

\[ F^h_i (\mathcal{Q}_s s^i) + \frac{1}{2} F^h_{ji} (\mathcal{Q}_s s^i) = 0, \]

(2.6)

then, for \( n>2 \), it defines an automorphism of the space, that is, the infinitesimal transformation \( v^h \) does not change both the metric and the almost complex structure of the space, and for \( n=2 \), it is contravariant almost analytic.

An almost Hermitian space in which \( F^h = 0 \) is satisfied is called an almost semi-Kählerian space. In such a space, we have
\[ F_{jik} F^{ik} = 2 F_t F^t_h = 0. \]

Thus from Theorem 2.2, we have

**Theorem 2.3.** If a conformal Killing vector \( v \) in an \( n (>2) \) dimensional compact almost semi-Kählerian space satisfies

(2.7) \[ F_{jik}(\nabla_i F^{ij}) = 0 \quad \text{or} \quad (\nabla_i F_{jik}) F^{it} = 0, \]

then \( v \) defines an automorphism in the space.

An almost Hermitian space in which \( F_{jik}=0 \) is satisfied is called an almost Kählerian space. In such a space, we have

\[ F_h = -\frac{1}{2} F_{jik} F^{ik} F_h^t = 0, \]

that is, \( F_{jik} \) is harmonic. Thus from Theorem 2.3, we have

**Theorem 2.4.** A conformal Killing vector \( v \) in an \( n (>2) \) dimensional compact almost Kählerian space defines an automorphism of the space (cf. [1; 2; 3]).

**Bibliography**


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