EXTENSION OF CONTINUOUS FUNCTIONS IN $\beta N$

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1. Introduction. The present considerations arose from the following problem: let $p \subseteq \beta N - N$; is $\beta N - N - \{p\}$ $C^*$-embedded in $\beta N - N$—i.e., is $\beta(\beta N - N - \{p\})$ equal to $\beta N - N$? We prove, assuming the continuum hypothesis (designated by \([CH]\)), that the answer is negative. More generally, see Theorem 4.6. The corresponding question for $\beta D$, where $D$ is any discrete space, is discussed in §5. The proofs depend upon results about $F$-spaces. We also prove \([CH]\) that all open subsets of $\beta R - R$ are $F$-spaces and that all open subsets of $\beta N - N$ are zero-dimensional $F$-spaces.

2. Background. All spaces considered are assumed to be completely regular. $N$ is the countably infinite discrete space, $R$ the space of reals. $C^*(X)$ denotes the ring of all bounded continuous functions from $X$ into $R$. A zero-set in $X$ is the set $Z(f)$ of zeros of a continuous function $f$. A cozero-set is the complement of a zero-set. Countable unions of cozero-sets are cozero-sets. A subspace $S$ of $X$ is $C^*$-embedded in $X$ if every function in $C^*(S)$ has a continuous extension to all of $X$. $\beta X$ denotes the Stone-Čech compactification of $X$, i.e., a compactification of $X$ in which $X$ is $C^*$-embedded.

The main results depend upon properties of $F$-spaces. Each of the following conditions characterizes $X$ as an $F$-space: any two disjoint cozero-sets are completely separated in $X$ (i.e., some function in $C^*(S)$ has a continuous extension to all of $X$). $\beta X$ is totally disconnected. We express these conditions by saying that $X$ is zero-dimensional. (For the requisite defini-

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$^3$ Symbols and terms are defined in §2. For additional details, see [2].
tion of dimension, see [2, Chapter 16].) If \( X \) is zero-dimensional, so is any \( C^* \)-embedded subspace.

3. Preliminary results. For any space \( Y \), \( vY \) denotes the set of all points \( p \) of \( \beta Y \) such that every zero-set in \( \beta Y \) that contains \( p \) also meets \( Y \). When \( vY = Y \), \( Y \) is said to be realcompact. All \( \sigma \)-compact spaces are realcompact [2, Chapter 8].

3.1. Lemma. If \( Y \) is locally compact and realcompact, then each zero-set in \( \beta Y - Y \) is the closure of its interior.

Remark. It suffices to prove that a nonempty zero-set \( Z \) has nonempty interior—for if \( p \in Z - \text{cl int } Z \), then some zero-set \( Z' \) disjoint from \( \text{int } Z \) contains \( p \), and \( \text{int}(Z \cap Z') \) is empty.

Proof. Since \( Y \) is locally compact, \( \beta Y - Y \) is compact and is therefore \( C^* \)-embedded in \( \beta Y \); hence \( Z = Z(f) - Y \) for some \( f \in C^*(\beta Y) \). Let \( p \in Z \). Since \( p \in vY \), there is a function \( g \) in \( C^*(\beta Y) \) that vanishes at \( p \) but nowhere on \( Y \). Define \( h = |f| + |g| \); then \( p \in Z(h) \subset Z \). Let \( (y_n) \) be a sequence of distinct points in \( Y \) on which \( h \) approaches 0. Choose disjoint compact neighborhoods \( V_n \) of \( y_n \) such that \( |h(y) - h(y_n)| < 1/n \) for \( y \in V_n \). It is easy to see that there exists a function \( u \in C^*(\beta Y) \) that is equal to 1 at each \( y_n \) and equal to 0 everywhere on \( Y - \bigcup_n V_n \). If \( q \) is any point of \( \beta Y - Y \) at which \( u(q) \neq 0 \), then every neighborhood of \( q \) meets infinitely many of the compact sets \( V_n \); hence \( h(q) = 0 \). Thus \( Z(h) \) contains the nonvoid open subset \( \beta Y - Y - Z(u) \) of \( \beta Y - Y \).

Local compactness is critical: an easy example shows that the conclusion of the lemma fails for \( \beta \mathbb{Q} - \mathbb{Q} \) (\( \mathbb{Q} \) = space of rationals).

3.2. Remark. If \( Y \) is locally compact and realcompact, but not compact, then \( \beta Y - Y \) is not basically disconnected—for 3.1 would imply that it is a \( P \)-space. (See [2] for definitions and for other proofs for \( \beta \mathbb{N} - \mathbb{N} \)).

3.3. Given \( X \), let \( S \subset X \) and \( p \in X - S \). The main results will be formulated in terms of the condition

\((p, S)\): There exist a neighborhood \( V \) of \( p \) and a cozero-set \( H \subset S \) such that \( S \cap V - H \) has void interior.

Trivially, \((p, S)\) holds whenever \( p \in \text{cl } S \); and if \( S \) is a cozero-set, then \((p, S)\) holds for every \( p \in S \).

3.4. Lemma. Let \( F \) be a compact set in \( X \) such that \((p, X - F)\) holds for all \( p \in F \). Then \( \text{int } Z \subset F \subset Z \) for some zero-set \( Z \).

Proof. There exist a finite open cover \( \{V_1, \ldots, V_n\} \) of \( F \) and zero-sets \( Z_1, \ldots, Z_n \) containing \( F \) such that \( \text{int } Z_k - F \subset X - V_k \). Let
$Z_0 = \bigcap_k Z_k$; then $Z_0$ is a zero-set containing $F$ and \( \text{int} Z_0 - F \subseteq X - \bigcup_k V_k \subseteq X - F \), so that $\text{cl}(\text{int} Z_0 - F)$ does not meet $F$. Since $F$ is compact, it is contained in a zero-set $Z'$ disjoint from $\text{int} Z_0 - F$ (see, e.g., [2, 3.11]). This implies that $\text{int}(Z_0 \cap Z') \subseteq F \subseteq Z_0 \cap Z'$.

3.5. Theorem. Let $X$ be an $F$-space, and let $S \subseteq X$ and $p \in \text{cl} S - S$. If $S \cap V$ is open for some neighborhood $V$ of $p$, and if $(p, S)$ holds, then $S$ is $C^*$-embedded in $S \cup \{p\}$.

Proof. There exist a neighborhood $V$ of $p$ and a cozero-set $H \subseteq S$ such that $S \cap V$ is open and is disjoint from $\text{int}(X - H)$. Given $f \in C^*(S)$, let $g \in C^*(H \cup \{p\})$ be an extension of $f|_H$ (see §2). Define $h$ on $S \cup \{p\}$ to agree with $f$ on $S$ and with $g$ at $p$. Since $H \cap V$ is dense in $(S \cup \{p\}) \cap V$, $h$ is a continuous extension of $f$.

4. The main results.

4.1. Theorem. Let $X$ be an $F$-space and let $S \subseteq X$ be a union of $\aleph_1$ cozero-sets $S_\alpha$ (in $X$). Then (a) $S$ is an $F$-space; (b) if $X$ is zero-dimensional, so is $S$; (c) if $G \subseteq S$ and $G \cap S_\alpha$ is a cozero-set in $S$ (for each $\alpha$), then $G$ is $C^*$-embedded in $S$.

Proof. (c). We may assume that $S = \bigcup_{\alpha < \omega_1} S_\alpha$ and that $S_\alpha \subseteq S_\beta \subseteq \cdots$. Notice that every $S_\alpha$ is an $F$-space. Let $g \in C^*(G)$ be given. Put $g_\xi = g|_{G \cap S_\xi}$. Given $\alpha < \omega_1$, assume that $g_\xi$ has been extended to $s_\xi \in C^*(S_\xi)$, for each $\xi < \alpha$, and that $s_{\alpha} \subseteq S_\beta \subseteq \cdots$. The function $U_{t < \alpha} s_t \cup g_\alpha$ is well defined and continuous on the cozero-set $U_{t < \alpha} S_t \cup (G \cap S_\alpha)$ in the $F$-space $S_\alpha$; hence it has an extension to a function $s_\alpha \in C^*(S_\alpha)$. Finally, $U_{\alpha < \omega_1} s_\alpha$ is a continuous extension of $g$ to all of $S$.

(a) If $G$ is a cozero-set in $S$, then by (c), $G$ is $C^*$-embedded.

(b) Completely separated sets in $S$ are contained in disjoint cozero-sets $A$ and $B$ in $S$. Let $g \in C^*(A \cup B)$ be equal to 0 on $A$ and to 1 on $B$. Note that every $S_\alpha$ is zero-dimensional. In the proof of (c), (with $G = A \cup B$), add to the induction hypothesis that $U_{t < \alpha} s_\xi$ is two-valued; then $s_\alpha$ may be taken to be two-valued.

4.2. Corollary. [CH]. All open subsets of $\beta \mathbb{R} - \mathbb{R}$ are $F$-spaces. All open subsets of $\beta \mathbb{N} - \mathbb{N}$ are zero-dimensional $F$-spaces.

Proof. Both $\beta \mathbb{R} - \mathbb{R}$ and $\beta \mathbb{N} - \mathbb{N}$ have just $\aleph_0$ zero-sets.

4.3. Theorem. [CH]. Let $X$ be an $F$-space having just $\aleph_0$ zero-sets. Let $S$ be open and let $p \in \text{cl} S - S$, and suppose that $(p, S)$ fails. Then (a) $S$ is not $C^*$-embedded in $S \cup \{p\}$; (b) $|\beta S - S| \geq \exp \exp \aleph_1$; (c) if $X$ is zero-dimensional, there is a two-valued function in $C^*(S)$ that has no continuous extension to $p$.

Proof. (a). Let $(S_t)_{t < \omega_1}$ be a family of cozero-sets in $X$ whose
union is $S$, and let $(V_\xi)_{\xi<\omega_1}$ be a base of zero-set-neighborhoods of $p$. Inductively, for each $\alpha<\omega_1$, assume that cozero-sets $A_\xi$ and $B_\xi$, contained in $S$, have been defined for all $\xi<\alpha$. Because $(p, S)$ fails, we can choose disjoint, nonempty cozero-sets $A_\alpha$ and $B_\alpha$ contained in $S \cap V_\alpha = \bigcup_{\xi<\alpha} (A_\xi \cup B_\xi \cup S_\xi)$.

Define $A = \bigcup_{\alpha<\omega_1} A_\alpha$, $B = \bigcup_{\alpha<\omega_1} B_\alpha$, and $G = A \cup B$. By construction, for each $\xi<\omega_1$, $G \cap S_\xi$ is the cozero-set $\bigcup_{\alpha<\xi} (A_\alpha \cup B_\alpha) \cap S_\xi$. By 4.1(c), $G$ is $C^*$-embedded in $S$. But $A$ and $B$ are complementary open sets in $G$ and each meets every neighborhood of $p$; therefore $G$ is not $C^*$-embedded in $G \cup \{p\}$. It follows that $S$ is not $C^*$-embedded in $S \cup \{p\}$.

(c) This now follows from 4.1(b).

(b) Since $|S| \leq \exp \exp |\mathbb{N}|$ (every point being an intersection of zero-sets), it is sufficient to show that $|\beta S| \leq \exp \exp |\mathbb{N}|$. Because $G$ is $C^*$-embedded in $S$, $|\beta S| \geq |\beta G|$. Clearly, $G$ contains a $C^*$-embedded copy of the discrete space $D$ of cardinal $|\mathbb{N}|$; so $|\beta G| \geq |\beta D|$. Finally, $|\beta D| = \exp \exp |\mathbb{N}|$, as is well known.

**4.4. Corollary [CH].** Let $X$ be an $F$-space with just $\exp |\mathbb{N}|$ zero-sets, and let $S \subseteq X$ be open and $p \in \text{cl } S - S$. Then $S$ is $C^*$-embedded in $S \cup \{p\}$ if and only if $(p, S)$ holds.

Proof. 3.5 and 4.3(a).

**4.5. Question.** Suppose that $X$ is zero-dimensional and that a dense subset $S$ of $X$ is not $C^*$-embedded in $X$; does there then exist a two-valued function in $C^*(S)$ with no continuous extension to $X$? It is easy to see that the answer is "yes" in case $S$ itself is zero-dimensional.

**4.6. Theorem [CH].** Let $K$ be a compact $F$-space of the form (2.1) that has just $\exp |\mathbb{N}|$ zero-sets. (E.g., $K = \beta \mathbb{N} - \mathbb{N}$ or $K = \beta \mathbb{R} - \mathbb{R}$.) Then:

(a) No proper dense subset is $C^*$-embedded—i.e., the equation $\beta X = K$ has the unique solution $X = K$.

(b) The following are equivalent for an open set $S$:

(i) $S$ is $C^*$-embedded in $K$.

(ii) $S$ is a cozero-set.

(iii) $(p, S)$ holds for all $p \in K - S$.

(c) If $S$ is open but is not a cozero-set, then $|\beta S| \geq \exp \exp |\mathbb{N}|$.

(d) If $K$ is totally disconnected (e.g., $K = \beta \mathbb{N} - \mathbb{N}$), and if $S$ is open but is not a cozero-set, then there is a two-valued function in $C^*(S)$ that has no continuous extension to all of $K$.

Proof. We prove first that (iii) implies (ii): by 3.1, $\text{cl } \text{int } Z = Z$ for every zero-set $Z$ in $K$; hence (iii) and 3.4 imply that $K - S$ is a zero-
set. Conclusions (b), (c), and (d) now follow from 4.3. Since no point of $K$ is isolated, 3.1 shows that no point is a zero-set; by (b), the complement of a point is not $C^*$-embedded, and this implies (a).

4.7. Remark. Let $X$ be a compact $F$-space; if $S \subseteq X$ and $|X - S| < \exp \exp N$, then $S$ is pseudocompact (i.e., every continuous function is bounded). For if $S$ admits an unbounded function, then $S$ contains $\mathbb{N}$ as a closed subset. Now, $N$ is $C^*$-embedded in $X$ [2, 14N] and so $\text{cl}_X N = \beta N$. But $X - S \supseteq \beta N - N$ and $|\beta N - N| = \exp \exp N$.

5. The space $\beta D - D$ for (uncountable) discrete $D$. If $A \subseteq D$ and $p \in \text{cl}_D A - D$, then $A - D$ is an open-and-closed neighborhood of $p$ in $\beta D - D$; these sets form a base at $p$ in $\beta D - D$. Let $E_0$ be the set of points of $\beta D - D$ in the closures of countable subsets of $D$, $E = \beta D - D - E_0$, $E_1$ the set of points of $E$ in the closures of subsets of $D$ of cardinal $N$. Then $E_0$ is countably compact and is open and dense in $\beta D - D$. Every compact subset of $E_0$ has an open neighborhood in $E_0$ homeomorphic with $\beta N - N$. Since $D$ is an $F$-space, so are $\beta D$ and its compact subspace $E$.

5.1. Theorem [CH]. If $p \in E_0$, then $\beta D - D - \{p\}$ is not $C^*$-embedded in $\beta D - D$.

Proof. $p$ has an open neighborhood in $E_0$ homeomorphic with $\beta N - N$, and 4.6(a) applies locally.

5.2. Theorem. If $S$ is an open subset of $E_0$, then either $S$ has compact closure in $E_0$ or $S$ has infinitely many limit points in $E_1$.

Proof. Let $\mathfrak{N}$ be a maximal family of disjoint, countably infinite subsets $N$ of $D$ for which $\text{cl} N - D \subseteq S$. Since $S$ is open, $\text{cl} \mathfrak{N} \supseteq S$. If $\mathfrak{N}$ is countable, then $\text{cl} \bigcup \mathfrak{N} - D \subseteq E_0$. If $\mathfrak{N}$ is uncountable, it has a subfamily $\mathfrak{N}'$ of cardinal $N$. Let $\mathfrak{F}$ be the filter on $D$ of all sets that contain all but finitely many points of $N$ for all but countably many $N \in \mathfrak{N}'$. Clearly, $\mathfrak{F}$ is contained in infinitely many (in fact, $\exp \exp N$) ultrafilters $\mathcal{U}$. For each such $\mathcal{U}$, consider $p = \lim \mathcal{U}$. Because the members of $\mathfrak{N}'$ are disjoint, every member of $\mathcal{U}$ is uncountable; hence $p \in E$. Since $\mathcal{U}$ contains the set $\bigcup \mathfrak{N}'$ of cardinal $N$, $p \in E_1$.

5.3. Lemma (Henriksen). If $p \in E - vD$; then $E - \{p\}$ is $C^*$-embedded in $E$.

Proof. $^6$ Since $p \in vD$, some function $f \in C^*(\beta D)$ vanishes at $p$ but

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$^4$ It is known that if $|D|$ is smaller than the first strongly inaccessible cardinal, then $vD = D$ (see §3).

$^6$ Due to Henriksen and Jerison; Henriksen's original proof was based on some results in the theory of lattice-ordered rings.
nowhere on \( D \). Every neighborhood of a point of \( E \) meets \( D \) in an uncountable set on which \(|f|\) is bounded away from zero. Hence \( E - Z(f) \) is a dense cozero-set in the \( F \)-space \( E \), and therefore the intermediate subspace \( E - \{ p \} \) is \( C^* \)-embedded in \( E \).

5.4. Theorem (Isbell-Jerison). If \( p \in E - vD \), then \( \beta D - D - \{ p \} \) is \( C^* \)-embedded in \( \beta D - D \).

Proof. Given \( g \in C^*(\beta D - D - \{ p \}) \), consider its restriction \( f = g|_{E - \{ p \}} \). By Henriksen's lemma, \( f \) can be extended continuously to \( p \)—say with the value 0 at \( p \). It suffices to show that \(|g|\) stays small near \( p \). Given \( \epsilon > 0 \), let \( V \) be an open-and-closed neighborhood of \( p \) such that \(|f(q)| < \epsilon \) for all \( q \in V \cap E \). Let \( S \) be the set of all points \( x \in V \cap E_0 \) such that \(|g(x)| > \epsilon \); then \( \text{cl} S \) meets \( E \) in at most the single point \( p \). To show that \( p \in \text{cl} S \), one may observe that \( S \) is open and apply 5.2. Thus, \(|g(x)| \leq \epsilon \) on the neighborhood \( V - \text{cl} S \) of \( p \).

5.5. Question. Is \( E_0 \) \( C^* \)-embedded in \( \beta D - D \)? If so, then \( E_0 \) is a zero-dimensional \( F \)-space. Note that 4.1 and [CH] yield the latter for the case \(|D| = \aleph_1 \). If \( E_0 \) is not \( C^* \)-embedded in \( \beta D - D \), then it is not \( C^* \)-embedded in \( D \cup E_0 \); this would imply that \( D \cup E_0 \) is not a normal space. In the case \(|D| = \aleph_1 \) (at least), it would also imply, by 4.5 and [CH], that some two-valued function in \( C^*(E_0) \) cannot be extended continuously to \( \beta D \).

We remark that the problem of extending two-valued functions from \( E_0 \) (for arbitrary \( D \)) can be formulated in the following way. Let \( \mathcal{Q} \) be the Boolean algebra of all subsets of \( D \), \( \mathcal{C} \) the subring of all countable subsets, and \( \mathcal{F} \) the ideal of all finite subsets. Let \( \Lambda \) be the set of all endomorphisms \( \lambda \) of \( \mathcal{C}/\mathcal{F} \) that satisfy (i): \( \lambda(x) \subseteq x \), and (ii): \( \lambda(\lambda(x)) = \lambda(x) \). Then the following are equivalent: every two-valued function in \( C^*(E_0) \) has a continuous extension to all of \( \beta D \); every \( \lambda \in \Lambda \) can be extended to \( \mathcal{Q}/\mathcal{F} \) so as to satisfy (i) and (ii).

References


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This is a modification of the Isbell-Jerison argument.