RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

POLYHEDRAL HOMOTOPY-SPHERES

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It has been conjectured that a manifold which is a homotopy sphere is topologically a sphere. This conjecture has implications, for example, in the theory of differentiable structures on spheres (see, e.g., [3, p. 33]).

Here I shall sketch a proof of the following theorem:

Let $M$ be a piecewise-linear manifold of dimension $n \geq 7$, which has the same homotopy-type as the $n$-sphere $S^n$. Then there is a piecewise-linear equivalence of $M - \{\text{point}\}$ with euclidean $n$-space; in particular, $M$ is topologically equivalent to $S^n$.

This theorem is not the best possible, for C. Zeeman has been able to refine the method presented here so as to prove the same theorem for $n \geq 5$.

A piecewise-linear $n$-manifold is a polyhedron with a linear triangulation satisfying the condition that the link of each vertex is combinatorially equivalent to the standard $(n-1)$-sphere; all the manifolds with which I am concerned here have no boundary. In general, all the spaces in this paper will be polyhedra, finite or infinite, and each map will be polyhedral, i.e., induced by a simplicial map of linear triangulations.

Let $K$ be a finite subpolyhedron of the finite polyhedron $L$; let $K'$ be a finite subpolyhedron of the finite polyhedron $L'$; let $f: L \rightarrow L'$ be a polyhedral map. $f$ is called a relative equivalence $(L, K) \Rightarrow (L', K')$, if $f(K) \subset K'$ and $L - K$ is mapped by $f$ in a 1-1 manner onto $L' - K'$.

Recall J. H. C. Whitehead's definition of contraction [7, p. 247]: If the simplicial complex $A$ has a simplex $\sigma_0$ which is the face of just one simplex $\tau_{p+1}$, and $B$ is the simplicial complex obtained from $A$ by removing the open simplexes $\sigma_0$ and $\tau_{p+1}$, then $A \rightarrow B$ is called an elementary contraction at $\sigma_0$. A finite sequence of elementary contractions is a contraction.

If $K$ is a finite subpolyhedron of the finite polyhedron $L$, then it is said that $L$ contracts onto $K$, if there is a linear triangulation $A$ of $L$.

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such that a subcomplex $B$ of $A$ triangulates $K$, and such that $A \to B$ is a contraction.

**Lemma 1.** If $L$ contracts onto $K$ and $(L, K)$ is a relative equivalence, then $L'$ contracts onto $K'$.

This can be shown by the methods of Whitehead [cf. 6, Theorem 1; 7, Theorem 6 and Theorem 7].

**Lemma 2.** Let $M$ be a piecewise-linear manifold, and let $L$ be a subpolyhedron which contracts onto $K \subseteq L$, and let $U$ be a neighborhood of $K$ in $M$. Then there is a piecewise-linear equivalence $h: M \to M$, such that $h(U)$ is a neighborhood of $L$ [cf. 7, Theorem 23].

The proof can be reduced to the case where $B \subseteq A \subseteq C$ are triangulations of $K \subseteq L \subseteq M$ respectively, and $A \to B$ is an elementary contraction at $\sigma^p$. It can then be reduced, due to the local euclidean nature of $M$ and the local nature of an elementary contraction, to the case when $M$ is euclidean space; the proof in this case is obvious.

An $n$-element $E$ is a polyhedron equivalent to the standard geometric $n$-simplex. Int $E$ will denote the subset of $E$ which corresponds to the interior of the $n$-simplex.

The following lemma was noticed, for $n = 3$, by Moise [4].

**Lemma 3.** Let the polyhedron $M$ contain two $n$-elements $E_1$ and $E_2$, such that $M = \text{Int } E_1 \cup \text{Int } E_2$. Then $M - \{\text{point}\}$ is piecewise-linearly equivalent to euclidean $n$-space; in particular $M$ is topologically a sphere.

This can be proved by the methods of B. Mazur [2] or M. Brown [1].

If $K$ is a finite polyhedron and $Q$ is the nonsingular join of $K$ to a point $x$, then $Q$ is called the cone on $K$ with vertex $x$. If $L$ is a finite subpolyhedron of $K$, then the cone $Q_l$ on $L$ with vertex $x$ is called a subcone of $Q$.

**Lemma 4.** A cone $Q$ contracts onto any subcone $Q_l$.

**Lemma 5.** If $P$ is a finite subpolyhedron of the cone $Q$, and dim $P \leq p$, then there is a subcone $Q_l$ of $Q$, such that $P \subseteq Q_l$ and dim $Q_l \leq p + 1$.

"dim" denotes dimension. The proofs of Lemmas 4 and 5 are elementary.

Let $K$ be a finite polyhedron, $M$ an $n$-manifold, and $f: K \to M$ a polyhedral map. Then $K$ is the union of a finite number of convex sets $\{\gamma_i\}$, on each of which $f$ is linear. $f$ is said to be in general position if there exists such $\{\gamma_i\}$ that for all $i, j$, 
(1) If \( \dim \gamma_i + \dim \gamma_j < n + \dim \gamma_i \cap \gamma_j \), then \( f|_{\gamma_i \cup \gamma_j} \) is 1-1;
(2) If \( \dim \gamma_i + \dim \gamma_j \geq n + \dim \gamma_i \cap \gamma_j \), then \( \dim (\gamma_i \cap f^{-1} \gamma_j) \leq \dim \gamma_i + \dim \gamma_j - n \).

The singular set of \( f: K \to M \), \( S(f) \), is the closure in \( K \) of the set \( \{ x \in K | f^{-1}(x) \) contains more than one point \}. The following lemma follows from property 2 of general position.

**Lemma 6.** If \( K \) is \( k \)-dimensional, and \( f: K \to M \) is in general position, where \( M \) is an \( n \)-dimensional manifold, then \( \dim S(f) \leq 2k - n \).

**Lemma 7.** If \( K \) is a subpolyhedron of \( L \), and \( M \) is a manifold, and \( f: L \to M \) is a map such that \( f|_{K} \) is in general position, then there is a map in general position \( g: L \to M \) such that \( g|_{K} = f|_{K} \).

The proof is obtained by localizing to the well-known proof for the case that \( M \) is a euclidean space (cf. [5]).

**Lemma 8 (Penrose-Whitehead-Zeeman [5]).** Let \( A \) be a subpolyhedron of the manifold \( M \), with \( 2(\dim A + 1) \leq \dim M = n \), and let \( A \) be contractible to a point in \( M \). Then \( A \) is contained in the interior of an \( n \)-element in \( M \).

The proof consists in embedding the cone on \( A, Q, \) in \( M \). There exists a map \( f: Q \to M \), such that \( f|_{A} = \text{inclusion} \); by Lemma 7, assume \( f \) is in general position. By Lemma 6, \( f \) will be nonsingular except in the case \( 2(\dim A + 1) = \dim M \), when there will be 0-dimensional singularities, which can be removed by a trick. Hence \( Q \subset M \); \( Q \) contracts to a point; a point in \( M \) is contained in the interior of an \( n \)-element. By Lemma 2, \( Q \) (and hence \( A \)) is contained in the interior of an \( n \)-element.

Let \( T \) be a linear triangulation of an \( n \)-manifold \( M \); \( T_p \) will denote the \( p \)-skeleton. \( T^* \) will denote the dual cell complex; \( T^*_q \) its \( q \)-skeleton.

**Lemma 9.** Let \( T \) be a linear triangulation of the \( n \)-manifold \( M \); let \( U \) and \( V \) be neighborhoods of \( T_p \) and \( T^*_q \) respectively, where \( p + q \geq n - 1 \). Then there is a polyhedral equivalence \( g: M \to M \), such that \( M = U \cup gV \).

This is proved by embedding \( M \) nicely in the join of \( T_p \) and \( T^*_q \) and applying a similar, elementary, lemma to that join.

**Proof of Theorem.**

(a) Case. \( n = 2k + 1, n \geq 7 \).

Let \( T \) be a linear triangulation of \( M \). Let \( Q \) be the cone on \( T_k \).
Let \( f: Q \to M \) be a map in general position such that \( f|_{T_k} \) is the inclusion of \( T_k \) into \( M \); such a map exists by Lemma 7 and the fact that \( M \) is \( k \)-connected.
By Lemma 6, since \( \dim Q = k + 1 \), and \( \dim M = 2k + 1 \), it follows that \( \dim S(f) \leq 1 \). By Lemma 5, there is a subcone \( Q_1 \subset Q \), such that \( S(f) \subset Q_1 \) and \( \dim Q_1 \leq 2 \).

By the theorem of Penrose, Whitehead, and Zeeman (Lemma 8), since \( \dim fQ_1 \leq 2 \) and \( \dim M \geq 6 \), there is an \( n \)-element \( E \subset M \) containing \( fQ_1 \) in its interior.

\( Q \) contracts into \( Q_1 \) (Lemma 4); since \( S(f) \subset Q_1 \), \( f \) defines a relative equivalence \((Q, Q_1) \Rightarrow (fQ, fQ_1)\); by Lemma 1, \( fQ \) contracts onto \( fQ_1 \). Since \( \text{Int} E \) is a neighborhood of \( fQ_1 \), by Lemma 2, there is a piecewise-linear homeomorphism \( h: M \to M \) such that \( fQ \subset h(\text{Int} E) \).

Hence \( \Delta_0 = h E \) is an \( n \)-element which is a neighborhood of \( fQ \). \( T_k \subset fQ \); hence \( \Delta_0 \) is a neighborhood of \( T_k \).

Similarly, an \( n \)-element \( \Delta_0^* \) may be found, which is a neighborhood of \( T_k^* \).

By Lemma 9, there is a piecewise-linear homeomorphism \( g: M \to M \) such that \( M = \text{Int} \Delta_0 \cup \text{Int} g\Delta_0^* \). Let \( \Delta_1 = g\Delta_0^* \). \( M \) is the union of the interiors of the two \( n \)-elements \( \Delta_0 \) and \( \Delta_1 \); hence by the Mazur-Brown Theorem (Lemma 3), the complement of a point of \( M \) is polyhedrally equivalent to euclidean \( n \)-space. In particular \( M \) is topologically a sphere.

(b) Case. \( n = 2k, n \geq 8 \).

The proof is very similar; the same notation is used. In this case, however, \( \dim S(f) \leq 2 \); \( \dim Q_1 \leq 3 \). The Penrose-Whitehead-Zeeman Theorem applies to \( fQ_1 \) since \( \dim fQ_1 \leq 3 \), and \( \dim M \geq 8 \). The rest of the proof is word for word the same.

**References**