A FAMILY OF SIMPLE GROUPS ASSOCIATED WITH
THE SIMPLE LIE ALGEBRA OF TYPE \((F_4)\)

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Communicated by Nathan Jacobson, September 12, 1960

In this note we obtain a family of simple groups, which also seem
to be new, by applying the method we used in \([3]\) to the Chevalley
groups of type \((F_4)\). The orders of the finite groups in the family are

\[
g^{12}(q - 1)^2(q + 1)(q^2 + 1)(q^3 + 1)(q^6 + 1),
\]

where \(q = 2^{2n+1}, \; n = 1, 2, 3, \ldots\).

Let \(\mathfrak{g}\) be the simple Lie algebra of type \((F_4)\) over the complex
number field, and \(\Sigma\) the root system of \(\mathfrak{g}\). Let the Coxeter-Dynkin
diagram of \(\Sigma\) be

\[
\begin{array}{cccc}
2 & 2 & 1 & 1 \\
\circ & \circ & \circ & \circ \\
a_1 & a_2 & a_3 & a_4
\end{array}
\]

Let \(P\) be the additive group generated by \(\Sigma\), and \(\phi: P \to P\) a homo-
morphism defined by (see \([2, \text{Exposé 24, p. 4}]\)),

\[
\begin{align*}
\phi(a_1) &= 2a_4, \\
\phi(a_2) &= 2a_3, \\
\phi(a_3) &= a_2, \\
\phi(a_4) &= a_1.
\end{align*}
\]

Then for any \(r \in \Sigma\) we have \(\phi(r) = \lambda(r)\bar{\tau}\), where \(\lambda(r)\)
is the length of the root \(r\) and where \(r \to \bar{\tau}\) is a permutation of order 2 of \(\Sigma\).

Let \(K\) be a field of characteristic 2 which admits an automorphism
\(t \to t^2\) such that \(2t^2 = 1\). Define the algebra \(g_K\) over \(K\) and the auto-
morphisms \(x_r(t)\), where \(r \in \Sigma, \; t \in K\), of \(g_K\) as in \([1]\), and let \(G\) be the
group generated by all the \(x_r(t)\). Then we have:

1. The group \(G\) admits an automorphism \(x \to x^\sigma\) such that

\[
x_r(t)^\sigma = x_r(\lambda^2(\bar{\tau}t^2))
\]

for all \(r \in \Sigma, \; t \in K\).

2. The group \(G^1\) of all elements \(x\) in \(G\) such that \(x = x^\sigma\) is simple
if \(K\) has more than two elements.

In order to describe the group \(G^1\) more closely, let \(\mathcal{U}\) be the sub-
group of \(G\) generated by all the \(x_r(t)\) with \(r > 0\), and set \(\mathcal{U}^1 = \mathcal{U} \cap G^1\). For \(r \in \Sigma, \; r > 0, \; \lambda(r) = 1\), set

\[
\alpha(t) = \begin{cases} 
x_r(t^2)x_r(t) & \text{if } r + \bar{\tau} \notin \Sigma, \\
x_r(t^2)x_r(t)x_{r+\bar{\tau}}(t^{2k+1}) & \text{if } r + \bar{\tau} \in \Sigma.
\end{cases}
\]

\[1\] This work was done while the author held a Research Associateship of the Office of Naval Research, U. S. Navy.

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Then $\alpha(t) \in \mathfrak{u}^1$, and from the 24 positive roots we obtain 12 such elements: $\alpha_1(t), \alpha_3(t), \cdots, \alpha_{12}(t)$. We have:

(3) Every element $x \in \mathfrak{u}^1$ is written uniquely as

$$x = \alpha_1(t_1)\alpha_2(t_2) \cdots \alpha_{12}(t_{12}),$$

where $t_i \in K$.

For any homomorphism $\chi: P \to K^*$ define $h(\chi) \in G$ as in [1]. Also see [1] for the meaning of the symbol $\omega(w)$, where $w \in W$, the Weyl group of $\Sigma$. We have:

(4) $h(\chi) \in G^1$ if and only if $\chi(a_i) = \chi(a_1)^g$, $\chi(a_3) = \chi(a_2)^g$.

(5) The group $W^1$ of all $w \in W$ such that $[w(r)]^* = w(r)$ for all $r \in \Sigma$ is of order 16, and for each $w \in W^1$ we can take $\omega(w)$ in $G^1$.

(6) Every element $x$ in $G^1$ is written uniquely as $x = uh(\chi)\omega(w)u'$, where: $u \in \mathfrak{u}^1$; $h(\chi) \in G^1$; $w \in W^1$ (we take $\omega(w)$ in $G^1$); $u'$ is a product of $\alpha(t)$ for which $w(r) < 0$.

If $K$ is a finite field of $q = 2^{2n+1}$ elements, where $n \geq 1$, then we can set $t^q = t^m$, where $m = 2n$. The order of $G^1$ can be computed from the above, since for each $w \in W^1$ we can find easily all $r \in \Sigma$ such that $r > 0$, $w(r) < 0$.

References