If $G = \{a, b, \cdots\}$ is a locally compact abelian group and $X = \{x, y, \cdots\}$ a complex commutative Banach algebra, we denote by $B(G, X)$ the generalized group algebra in the sense of [1; 2]. An $X$-valued function $g$ defined over $G$ is in $B(G, X)$ if $g$ is strongly measurable and Bochner integrable with respect to Haar measure over $G$. We define $\|g\|_{B(G, X)} = \int_{G} |g(a)| x \, da$ and, with convolution as multiplication, $B(G, X)$ is a complex commutative $\mathbb{C}$-algebra. In [1, p. 1606], it is shown that the space $\mathfrak{M}(B)$ of regular maximal ideals in $B(G, X)$ is homeomorphic with $\hat{G} \times \mathfrak{M}(X)$. Here, $\hat{G} = \{\hat{a}, \hat{b}, \cdots\}$ is the character group of $G$ and $\mathfrak{M}(X)$ denotes the space of regular maximal ideals in $X$, both in their usual topologies. If $\phi_M$ is the canonical homomorphism of $X$ onto the complex numbers associated with an $M \in \mathfrak{M}(X)$, then a function $g \in B(G, X)$ is represented on its space of maximal ideals $\mathfrak{M}(X)$ as $g_\phi = \phi_M g$ where $f \in L(G)$ and $\phi$ is a function defined on $\mathfrak{M}(X)$, then $g = fx$ for some $x \in X$. Clearly $fx \in B(G, X)$. Further, finite linear combinations of functions of the type $fx$ with $f \in L(G), x \in X$ are dense in $B(G, X)$.

In this paper we propose to characterize the homomorphisms $T$ from $B(G, X)$ into $B(G, X')$ which are such that $T$ keeps $L(G)$ “pointwise invariant.” More precise statements will be found in the theorems below.

We begin with

**Theorem 1.** Let $G$ be a group such that $\hat{G}$ is connected and let $X$ and $X'$ be commutative $\mathbb{C}$-algebras with identities $e, e'$ respectively. Suppose $\mathfrak{M}(X)$ is totally disconnected and $X'$ is semi-simple. Let $T: B(G, X) \to B(G, X')$ be a continuous homomorphism such that $T(f e) = f e'$ for any $f \in L(G)$. Then there exists a continuous homomorphism $\sigma: X \to X'$ such that $(Tg)(a) = \sigma g(a)$ for any $g \in B(G, X)$.

**Proof.** If $g' \in B(G, X')$ and $g'$ is represented on its space of maximal ideals $\hat{G} \times \mathfrak{M}(X')$ as $j \cdot \phi'$ where $f \in L(G)$ and $\phi'$ is a function defined on $\mathfrak{M}(X')$, then $g' = fx'$ for some $x' \in X'$. (Here, $\int_{\hat{G}} f(\hat{a})(a, \hat{a}) \, da.$) For, consider the function $F$ from $G$ to $X'$ given

\[ F(a) = \int_{\hat{G}} f(\hat{a})(a, \hat{a}) \, da. \]
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by \( F(a) = \int a g'(a)(a, \alpha) da \). If \( M' \in \mathfrak{M}(X') \), then \( \phi_{M'}(F(\alpha)) = \int \phi(M') \).

If \( f \neq 0 \), there exists an \( \alpha \) such that \( \int \phi(M') \neq 0 \). Let \( \alpha' = F(\alpha) / \int \phi(M') \) so that \( \phi_{M'}(x') = \phi'(M') \). We see that \( g' \) and \( f x' \) are represented by the same function on \( \hat{G} \times \mathfrak{M}(X') \). Now, since \( X' \) is semi-simple, \( B(G, X') \) is semi-simple [1, p. 1609] and thus \( g' = f x' \).

For \( \alpha \in \hat{G} \), \( M' \in \mathfrak{M}(X') \), we have in

\[
[T_{g'}(\alpha, M')] = \int a \phi_{M'}(T_{g'})(a)(a, \alpha) da
\]
a continuous multiplicative linear functional on \( B(G, X) \). This means that \( [T_{g'}]^{-1}(\alpha, M') \) is equal to \( \tau(\alpha) \), \( \sigma^*M' \) for some \( \tau(\alpha) \in \hat{G} \), \( \sigma^*M' \in \mathfrak{M}(X) \). Thus, \( T^*: \hat{G} \times \mathfrak{M}(X') \to \hat{G} \times \mathfrak{M}(X) \) given by \( T^*(\alpha, M') = (\tau(\alpha), \sigma^*M') \). Since \( \hat{G} \) is connected and \( T^* \) is continuous, \( T^*(\hat{G} \times \{ M' \}) \) is a connected set in \( \hat{G} \times \mathfrak{M}(X) \) for each \( M' \in \mathfrak{M}(X') \). Since \( \mathfrak{M}(X) \) is totally disconnected, \( T^*(\hat{G} \times \{ M' \}) = \hat{G} \times \{ \sigma^*M' \} \). This is true because the connected components of \( \hat{G} \times \mathfrak{M}(X) \) are precisely of the form \( \hat{G} \times \{ M \} \) with \( M \in \mathfrak{M}(X) \). Since \( T(fe) = fe', f \in L(G) \), we conclude that \( \tau(\alpha) = \alpha \) and \( T^* \) is the product of the identity map on \( \hat{G} \) and a map \( \sigma^*: \mathfrak{M}(X') \to \mathfrak{M}(X) \).

Consider \( f x \in B(G, X), f \in L(G) \). It gets represented as a product function on \( \hat{G} \times \mathfrak{M}(X) \). From the nature of \( T^* \) in the preceding paragraph, \( T(f x) \) gets represented as a product function on \( \hat{G} \times \mathfrak{M}(X') \) whose first factor is \( f(\alpha) \). In view of the second paragraph of this proof, there exists \( \sigma(x) \in X' \) such that \( T(f x) = \sigma(x)f \). The map \( \sigma: X \to X' \) is a continuous homomorphism as is easy to verify. As already remarked, finite linear combinations of functions \( f x \) are dense in \( B(G, X) \) and since \( T \) is continuous, the theorem is proved.

In our next theorem \( G \) will be taken compact and we require the following

**Lemma.** Suppose \( G \) is a compact abelian group with Haar measure normalized to 1, and \( X \) is a complex commutative \( B \)-algebra with identity \( e \) with no restrictions on \( \mathfrak{M}(X) \). Let \( \phi \) be a continuous homomorphism from \( B(G, X) \) to \( L(G) \) which is such that \( \phi(fe) = f \) for all \( f \in L(G) \). Then there exists an \( M \in \mathfrak{M}(X) \) such that \( (\phi g)(a) = \phi_M g(a) \) a.e. for any \( g \in B(G, X) \).

**Proof.** Since \( G \) is compact, the constant \( X \)-valued functions are in \( B(G, X) \) and thus \( X \) can be considered to be a subset of \( B(G, X) \). In other words, if \( x \in X \), we denote the function \( f(a) = x \) (for almost all \( a \in G \)) simply by \( x \) itself. If \( x, y \in X \subset B(G, X) \), then \( x * y = \int xy a = xy \) since \( m(G) = 1 \). (Here, \( xy \) denotes the ordinary product of \( x \) and \( y \) in the \( B \)-algebra \( X \).) Now \( \phi \) is not identically zero on \( X \) because
\( \phi(1 \cdot e) = 1 \). Further, for any \( x \in X \), \( \phi(x \cdot e) = \phi(xe) = \phi(x) = \phi(x) \cdot \phi(e) = \phi(x) \cdot 1 f. \phi(x) da = \text{a constant a.e. over } G \). Hence each \( x \in X \) is mapped by \( \phi \) onto a constant function in \( L(G) \). \( \phi \) is additive on \( X \) and furthermore: \( \phi(x \cdot y) = \phi(xy) = \phi(x) \cdot \phi(y) = \phi(x) \phi(y) \) for any \( x, y \in X \). Consequently \( \phi \) is a continuous nonzero multiplicative linear functional on the \( B \)-algebra \( X \) and, as such, there exists an \( M \subseteq \mathcal{M}(X) \) with \( \phi(x) = \phi_M(x) \) for \( x \in X \).

Choose an arbitrary \( f \in L(G) \), \( x \in X \) and \( \delta \in \mathcal{G} \). We have:
\[
\phi(f(x \cdot (\cdot, \delta)^{-1}e)) = f(\delta)\phi((\cdot, \delta)^{-1}x) = f(\delta)\phi((\cdot, \delta)^{-1}x) = \phi(fx) \cdot \phi((\cdot, \delta)^{-1}e) = \phi(fx) \cdot (\cdot, \delta)^{-1} = (\cdot, \delta)^{-1}[\phi(fx)](\delta). \]
Hence, for each \( \delta \in \mathcal{G} \),
\[
[\phi(fx)](\delta) = (a, \delta)f(\delta)\phi((a, \delta)^{-1}x) = (a, \delta)(a, \delta)^{-1}f(\delta)\phi(x) = f(\delta)\phi_M(x). \]
This implies \( \phi(fx) = \phi_M(x)f \) for all \( f \in L(G) \), \( x \in X \).

Taking finite linear combinations of functions of the type \( fx \) with \( f \in L(G) \) and \( x \in X \), we can find a sequence \( \{f_n\} \) such that \( f_n \rightarrow g \) for any \( g \in B(G, X) \) with \( \phi(f_n) = \phi_M(f_n) \). Hence \( \phi(g) = \phi_M(g) \) since \( \phi \) is continuous and the lemma is established.

**Theorem 2.** Let \( G \) and \( X \) be as in the lemma and let \( X' \) denote a semi-simple \( B \)-algebra with identity \( e' \). Suppose \( T : B(G, X) \rightarrow B(G, X') \) is a continuous homomorphism such that \( T(fe) = fe' \) for any \( f \in L(G) \). Then there exists a continuous homomorphism \( \sigma : X \rightarrow X' \) such that \( (Tg)(\sigma) = \sigma g(\sigma) \) for any \( g \in B(G, X) \).

**Proof.** Let \( \{W\} \) be the set of neighborhoods of the identity \( 0 \in G \) and let \( \{j_w\} \) be an approximate identity in \( L(G) \). Then, if \( f \in L(G) \), \( x \in X \), we have \( T(j_w \cdot fx) = T(j_w \cdot fx) \cdot fe' \rightarrow T(fx) \) as \( W \rightarrow 0 \). Taking Fourier transforms we find
\[
[T(j_w \cdot fx)](\delta) \rightarrow [T(fx)](\delta) \quad \text{so that} \quad [T(j_w \cdot fx)](\delta) \text{ converges as } W \rightarrow 0 \text{ for each } \delta \in \mathcal{G}. \]
Call this limit \( \sigma_d(x) \). It is clear that \( \sigma_d(x) \) is independent of \( \delta \) and the function \( \sigma \) defining it.

Let \( M' \subseteq \mathcal{M}(X') \) and consider the map \( \phi_{M'} \circ T \) from \( B(G, X') \) to \( L(G) \). If \( fe \in B(G, X) \) with \( f \in L(G) \), then \( (\phi_{M'} \circ T)(fe) = \phi_{M'}(fe') = f \). The map \( \phi_{M'} \circ T \) is continuous and the lemma applies to it. Therefore, there is an \( M \subseteq \mathcal{M}(X) \), depending on \( M' \subseteq \mathcal{M}(X') \), such that \( \phi_{M'} \circ T = \phi_M \). Now:
\[
\phi_{M'}[T(fx)](\delta) = [\phi_{M'} \circ T](fx)](\delta) = f(\delta)\phi_M(x) = f(\delta)\phi_{M'}(\sigma_d(x)). \]
This means \( \phi_{M'}(\sigma_d(x)) = \phi_M(x) \) for each \( M' \subseteq \mathcal{M}(X') \) and each \( \delta \in \mathcal{G} \), \( x \in X \).

We show that \( \sigma_d(x) \) is actually independent of \( \delta \). Suppose \( \sigma_{d_1}(x) = y_1 \), \( \sigma_{d_2}(x) = y_2 \) and \( d_1 \neq d_2 \). For any \( M' \subseteq \mathcal{M}(X') \) we have \( \phi_{M'}(y_1) = \phi_{M'}(y_2) = \phi_M(x) \). Since \( X' \) is semi-simple, we must have \( y_1 = y_2 \) and
so \( \sigma_\delta(x) \) is independent of \( \delta \). Write \( \sigma_\delta(x) = \sigma(x) \). \( \sigma \) is a continuous homomorphism from \( X \) to \( X' \). We have, consequently, shown that 
\[
[T(fx)](\delta) = \sigma(x)f(\delta)
\]
d and this means \( T(fx) = \sigma(x)f \) for all \( f \in L(G) \), \( x \in X \), because \( B(G, X') \) is semi-simple if \( X' \) is semi-simple. Continuing in a manner like that at the end of the lemma or the end of Theorem 1, we find that \( (Tg)(a) = \sigma g(a) \) for all \( g \in B(G, X) \). This completes the proof.

We remark that, conversely, if \( \sigma: X \to X' \) is a continuous homomorphism, then the map \( (Tg)(a) = \sigma g(a) \) from \( B(G, X) \) to \( B(G, X') \) is a continuous homomorphism with no restrictions on \( G, \hat{G}, X \) or \( X' \). The proof is easy and is omitted.

**Theorem 3.** In either Theorem 1 or 2, if \( T \) is an isomorphism from \( B(G, X) \) onto \( B(G, X') \), then \( \sigma \) is an isomorphism from \( X \) onto \( X' \).

**Proof.** \( \sigma \) is one-one for if \( x_1 \neq x_2, x_1, x_2 \in X \), then \( fx_1 \neq fx_2 \) where \( f \in L(G), f \neq 0 \). Since \( T \) is one-one, \( T(fx_1) = \sigma(x_1)f \neq T(fx_2) = \sigma(x_2)f \) so that \( \sigma(x_1) \neq \sigma(x_2) \). \( \sigma \) is onto \( X' \), for choose any \( x' \in X' \). Find an \( f \in L(G) \) such that \( f'(\hat{0}) \neq 0 \). Since \( T \) is onto, there is a \( g \in B(G, X) \) such that \( \sigma g = fx' \). Taking Fourier transforms: 
\[
[\sigma g](\delta) = \sigma g(\delta) = \hat{f}(\delta)x'.
\]
Setting \( \delta = \hat{0} \), we find \( \sigma g(\hat{0}) = \hat{f}(\hat{0})x' \) so that \( \sigma(\hat{g}(\hat{0})/\hat{f}(\hat{0})) = x' \). Hence, there is an \( x = \hat{g}(\hat{0})/\hat{f}(\hat{0}) \in X \) such that \( \sigma(x) = x' \).

**References**


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