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**SUMMABILITY (L) OF FOURIER SERIES**

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Communicated by S. Bochner, September 28, 1960

1. In a recent paper, Borwein [1] has constructed a new method of summability for an infinite sequence \((S_n)\). He defined a sequence \((S_n)\) to be summable by the logarithmic method of summability or summable \((L)\) to the sum \(s\) if, for \(x\) in the interval \((0,1)\)

\[
\lim_{x \to 1^-} \frac{1}{|\log(1 - x)|} \sum_{n=1}^{\infty} \frac{S_n}{n} x^n = s,
\]

which is written simply as \(S_n \to s (L)\). Concerning this kind of summability, Borwein has established a number of fundamental facts. For instance, he showed that \((L) \supset (A, \lambda). \)\(^1\) Thus, we have the following full inclusive relation:

\[(L) \supset (A, \lambda) \supset (A) \supset (C, r),\]

for any \(r > -1\), where \((A)\) is the ordinary Abel's summability and \((C, r)\) is the Cesàro summability of order \(r\).

In this note, the author intends to apply this new method of summability to the Fourier series of \(f(x)\) in order to obtain a corresponding summability criterion for it.

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1 A sequence \((S_n)\) is said to be summable \((A, \lambda)\) to the sum \(s\) if \((1 - x) \sum S_n x_n \to s\) as \(x \to 1^-\), where \(S_n\) is the \(n\)th Cesàro mean of order \(\lambda\) of \((S_n)\) [1, p. 212 and §3, Theorem 3].
2. Suppose that \( f(x) \) is a Lebesgue integrable function, periodic with period \( 2\pi \). Let
\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
be its Fourier series. Fixing \( x_0 \), we write
\[
\phi(t) = \phi_{x_0}(t) = \frac{1}{2} \{f(x_0 + t) + f(x_0 - t) - 2s\}.
\]

First, we derive the following fundamental theorem concerning the kernel of the summability \((L)\) for Fourier series.

**Theorem 1.** The necessary and sufficient condition for the Fourier series of \( f(x) \) to be summable \((L)\) to the sum \( s \) at the point \( x_0 \) is that
\[
\int_{0}^{\pi} \phi(t) \tan^{-1} \frac{x \sin t}{1 - x \cos t} \, dt = o(\left| \log(1 - x) \right|)
\]
as \( x \to 1 - 0 \).

Let
\[
S_n(x_0) = \frac{1}{2} a_0 + \sum_{r=1}^{n} (a_r \cos nx_0 + b_r \sin nx_0)
\]
be the \( n \)th partial sum of the Fourier series of \( f(x) \) at \( x_0 \). Then, we have
\[
S_n(x_0) - s = \frac{1}{\pi} \int_{0}^{\pi} \phi(t) \frac{\sin nt}{t} \, dt + o(1).
\]

Thus,
\[
\sum_{n=1}^{\infty} \frac{1}{n} \{S_n(x_0) - s\} x^n
\]
\[
= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{x^n}{n} \int_{0}^{\pi} \phi(t) \frac{\sin nt}{t} \, dt + o\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right)
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \phi(t) \left(\sum_{n=1}^{\infty} \frac{\sin nt}{n} x^n\right) dt + o(\left| \log(1 - x) \right|)
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \phi(t) \tan^{-1} \frac{x \sin t}{1 - x \cos t} \, dt + o(\left| \log(1 - x) \right|).
\]

Now,
\[
\sum_{n=1}^{\infty} \frac{1}{n} \{ S_n(x_0) - s \} x^n = \sum_{n=1}^{\infty} \frac{S_n(x_0)}{n} x^n - s \log(1 - x) \]
\[
= L(x) - s \log(1 - x).
\]

Hence, the sequence \((S_n(x_0))\) is summable \((L)\) to \(s\) if and only if

\[
\int_0^x \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} \, dt = o(\log(1 - x))
\]
as \(x \to 1 - 0\). This establishes the theorem.

3. Next, we derive a summability criterion of \((L)\) summability for the Fourier series of \(f(x)\) at \(x_0\) as follows.

**Theorem 2.** If

(i) \(\int_0^t |\phi(u)| \, du = o(t \log t)\), \((t \to + 0)\),

(ii) \(\int_t^\delta (|\phi(u)| /u)\, du = o(\log t)\),

as \(t \to +0\) for any arbitrary \(0 < \delta < \pi\), then the Fourier series of \(f(x)\) is summable \((L)\) to \(s\) at \(x_0\).

For, if we write

\[
\int_0^x \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} \, dt = \int_0^{1-z} + \int_{1-z}^\delta + \int_\delta^\pi
\]

\[
= J_1(x) + J_2(x) + J_3(x),
\]
say. Then, since

\[
\lim_{t \to +0} \frac{1}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} = \frac{x}{1 - x},
\]

we can choose \(x_0\) sufficiently near \(1\) such that

\[
|J_1(x)| < \frac{2x}{1 - x} \int_0^{1-z} |\phi| \, dt
\]

for \(0 < x_0 < x < 1\). It follows that \(J_1(x) = o(\log(1 - x))\) as \(x \to 1 - 0\) by (i). Considering that

\[
|\tan^{-1} \frac{x \sin t}{1 - x \cos t}| < \frac{\pi}{2}
\]
uniformly for $0 \leq x < 1$ and $0 < t \leq \pi$, we find

$$|J_s(x)| < \frac{\pi}{2} \int_{1-x}^{1} \frac{|\phi|}{t} \, dt = o(|\log(1 - x)|)$$

as $x \to 1 - 0$ by (ii). Last, we have

$$|J_s(x)| = \left| \int_{s}^{\pi} \frac{\phi}{t} \tan^{-1} \left( \frac{x \sin t}{1 - x \cos t} \right) \, dt \right|$$

$$\leq \frac{1}{\delta} \int_{s}^{\pi} \phi \left| \tan^{-1} \left( \frac{x \sin t}{1 - x \cos t} \right) \right| \, dt$$

$$< \frac{\pi}{2\delta} \int_{0}^{\pi} \phi \, dt$$

$$= O(1)$$

$$= o(|\log(1 - x)|)$$

as $x \to 1 - 0$. This proves Theorem 2.

4. Accordingly, from the estimation of $J_s(x)$ in the proof of the above theorem, we get the following almost self-evident

**Theorem 3.** The $(L)$ summability of the Fourier series of $f(x)$ at $x_0$ is a local property of $f(x)$ near $x_0$. I.e.,

$$L(x) = \frac{1}{\pi} \int_{0}^{\delta} \frac{\phi(t)}{t} \tan^{-1} \left( \frac{x \sin t}{1 - x \cos t} \right) \, dt + o(|\log(1 - x)|)$$

for any arbitrary $0 < \delta < \pi$ as $x \to 1 - 0$.

**References**


**National Taiwan University**