RESEARCH PROBLEMS


Consider the equation

\[ u = 1 + T(xu), \]

where \( T \) is a linear transformation and \( x \) and \( u \) are elements of a suitable space. As pointed out by Baxter, if \( T \) enjoys the following generalized "integration by parts" formula

\[ T(xT(y)) + T(yT(x)) = T(x)T(y) \]

then \( u = e^{T(x)} \) is a solution of (1), and the solution under appropriate assumptions.

Generally, let us write \( u = E(x) \) to denote the solution of (1), a generalized exponential, and introduce a generalized bracket symbol

\[ [x, y; T] = T(xT(y)) + T(yT(x)) - T(x)T(y). \]

It is easy to verify that this enjoys the Friedrichs-Magnus property

\[ [x + x', y + y'; T] = [x, y; T] + [x', y'; T]. \]

One would suspect in view of the foregoing remarks that this new bracket symbol plays the same role in the study of the function \( E(x) \) that the classical commutator, \( [A, B] = AB - BA \), plays in the study of the matrix exponential.

We would thus expect formulas of the type

\[ E(x + y) = E(x)E(y)E([x, y; T]) \cdots, \]

where the further terms contain iterations of the bracket operation, and equivalently, a generalized Baker-Hausdorff-Campbell formula

\[ E(x)E(y) = E(x + v + [x, y; T]/2 + \cdots). \]

Finally, there should be an associated generalized Lie algebra.

Do formulas of the above type exist, and how does one obtain them?


Let \( Z^n \) denote the set of lattice points in euclidean \( n \)-space, \( I^n \) the set of points in \( Z^n \) all of whose coordinates are 0 or 1 (i.e., the vertices of the unit cube). Let \( > \) be a total order on \( I^n \) which is "consistent" in the sense that for \( x, y, z, x+z, y+z \in I^n \), \( x > y \) implies \( x+z > y+z \) (ordinary vector addition is meant). Is it always possible to extend \( > \) to a "consistent" order on \( Z^n \)?