INTEGRATION WITH RESPECT TO OPERATOR-VALUED FUNCTIONS

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1. Introduction. Let \( J \) be a compact subinterval of the real line. N. Wiener [7] has introduced the Banach algebra \( W_p(J) \) of all complex-valued functions \( f \) such that
\[
V_p(f) = \sup \left( \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p \right)^{1/p},
\]
the supremum being taken over all finite partitions of \( J \) (see §7). We shall construct a family of continuous homomorphisms of the Banach algebra \( W_p(J) \); this connects with the theory of multipliers of Fourier series (see §4). Our basic problem is to integrate (in the uniform operator-topology) with respect to functions that are not of bounded variation.

Given a fixed measurespace \((a, \mathfrak{A}, \mu)\), let \( \mathcal{E}_r \) denote the Banach algebra of all continuous endomorphisms of \( L_r(a, \mathfrak{A}, \mu) \) ; the relation \( 1 < r < \infty \) is implied throughout. Let \( E_r \) be a function on \( J \) which assumes its values in \( \mathcal{E}_r \), and let \( f \) belong to the class \( D(J) \) of all simply-discontinuous,\(^2\) complex-valued functions. The following expression
\[
(1) \quad (\mathcal{E}_r) \int f(\lambda) \cdot dE_r(\lambda)
\]
will denote what T. H. Hildebrandt [1, p. 273] calls the "modified Stieltjes integral"; it is the limit of a certain net of Stieltjes sums (this net is directed as in the Pollard-Moore integral [1, p. 269]). The word "limit" here implies convergence in the norm-topology of \( \mathcal{E}_r \). It is not hard to show that the integral (1) converges when \( E_r \) is of bounded variation\(^3\); this situation is most familiar in the case \( r = 2 \), when \( E_r \) is a resolution of the identity in the Hilbert space \( L_2(a, \mathfrak{A}, \mu) \). Henceforth, we will allow the possibility that \( E_r \) not be of bounded variation (this possibility becomes a fact in Theorem D below).

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2 That is, having on \( J \) at most discontinuities of the first kind.

3 In the sense of Hille-Phillips [2, p. 59]. Bounded variation is a less restrictive condition than the bounded semi-variation hypothesis required in certain integration theories (e.g., Bartle's article in the Studia Math. vol. 15 (1956) pp. 337–352).
2. **Motivation.** Suppose that \( r \neq 2 \). Some integrators \( E_r \) have the following property: there exists no spectral measure \( M \) such that

\[
\int \lambda \cdot M(d\lambda) = (E_r) \int \lambda \cdot dE_r(\lambda),
\]

although the integral on the right-hand side converges.

3. **An operational calculus.** Let \( L^0(\alpha, \beta, \gamma) \) be the class of all simple functions. If \( T \in \mathcal{E}_2 \) we write

\[
|T|_r = \sup \{ \|Tx\|_r : x \in L^0(\alpha, \beta, \gamma) \text{ and } \|x\|_r \leq 1 \};
\]

it is clear that the eventuality \( |T|_r \neq \infty \) implies the existence of the continuous extension (denoted \( T_r \)) of \( T \) from \( L^0(\alpha, \beta, \gamma) \) to \( L_r(\alpha, \beta, \gamma) \).

Suppose that \( E \) is a resolution of the identity in \( L_r(\alpha, \beta, \gamma) \) such that

\[
|E(\lambda)|_r \neq \infty \text{ whenever } 1 < s < \infty.
\]

If \( \lambda \in J \), then \( E(\lambda) \in \mathcal{E}_2 \) and \( |E(\lambda)|_r \neq \infty \), whence \( E(\lambda)_r \in \mathcal{E}_r \) (here \( E(\lambda)_r \) again denotes the extension of \( E(\lambda) \) from \( L^0(\alpha, \beta, \gamma) \) to \( L_r(\alpha, \beta, \gamma) \)). Accordingly, we may define on \( J \) a function \( E_r \) by means of the relation \( E_r(\lambda) = E(\lambda)_r \). Finally, let \( I(p) \) denote the open interval with endpoints \( 2p/(p \pm 1) \). Under these circumstances, it can be proved that:

*if \( 1 \leq p < \infty \) and \( r \in I(p) \), then each integral of the family

\[
\left\{(E_r) \int f(\lambda) \cdot dE_r(\lambda) : f \in W_p(J) \right\}
\]

converges in the norm-topology of \( \mathcal{E}_r \).

It is notable that, if \( \infty > p > q > 1 \), then

\[
\{2\} \subset I(p) \subset I(q) \subset I(1) = \{\lambda : 1 < \lambda < \infty\},
\]

\[
D(J) \supset W_p(J) \supset W_q(J) \supset W_1(J) = \{\text{bounded variation}\}.
\]

In other words: as the range of \( r \) expands from the Hilbert-space case \( \{2\} \) to comprise the whole interval \((1, \infty)\), then \( W_p(J) \) contracts into the class \( W_1(J) \) consisting of all functions of bounded variation.

There exists a well-known bijection \( \{ T \mapsto E^T \} \) of the class \( \mathcal{E} \) (of all self-adjoint members of \( \mathcal{E}_0 \)) into the class of all resolutions of the identity \([6, \text{p. 174 and p. 176}]\). Suppose \( T \in \mathcal{E} \), and let \( J \) be an interval that contains the spectrum of \( T \). It will be convenient to write

\[
f(T_r) = (E_r) \int f(\lambda) \cdot dE_r^T(\lambda).
\]
An application of the Spectral Theorem shows easily that:

\[ f(\lambda) = \sum_n \alpha_n \cdot \lambda^n \text{ is a polynomial, then } f(T_\lambda) = \sum_n \alpha_n \cdot (T_\lambda)^n. \]

**Theorem A.** Suppose that \( T \in \mathfrak{H} \) and let condition (v) be satisfied when \( E \) is replaced by \( E^r \). If \( 1 \leq p < \infty \) and \( r \in I(\phi) \), then the mapping \( \{ f \to f(T_\lambda) \} \) is a continuous homomorphism of the Banach algebra \( W_p(J) \) into \( \mathfrak{C}_r \).

4. Two applications to the theory of multiplier transformations.

Consider a complete orthonormal system \( \{ \Phi_n : n \in a \} \); accordingly, \( a \) is denumerable. In this paragraph, \( a \) consists of all subsets of \( a \), while \( \mu \) is taken to be counting-measure; thus \( L_\mu(a, \alpha, \mu) \) becomes the sequence space usually denoted \( l_\alpha \), and \( L_\mu^0(a, \alpha, \mu) \) is now the class \( l_0^0 \) of all sequences that vanish off finite subsets of \( a \). If \( x \in l_0^0 \), then \( f^*(x) \) will denote the sequence of Fourier coefficients of the function \( f \cdot x^\sim \), where

\[ (f \cdot x^\sim)(\lambda) = f(\lambda) \cdot \sum_{n \in a} x_n \cdot \Phi_n(\lambda) \quad (\lambda \in J). \]

Let \( f^* \) be the mapping \( \{ x \to f^*(x) \} \) defined on \( l_0^0 \). Hirschman [3] calls \( f^* \) a "multiplier transformation." An important problem in the theory of multiplier transformations is to find conditions on \( f \) which will insure that \( |f^*| \neq \infty \).

**First application.** Let \( \{ \Phi_n : n \in a \} \) be the system of normalized Legendre polynomials on \( J = [-1, 1] \), and denote by \( T \) the member \( \Delta \) of \( \mathfrak{H} \) that is defined in [4, (2)]. The article [4] shows that condition (v) is satisfied when \( E \) is replaced by \( E^r \). Suppose \( 1 \leq p < \infty \) throughout. Our theory shows that

(i) \[ \text{if } r \in I(\phi) \text{ and } f \in W_p(J), \text{ then } |f^*| \neq \infty; \]

consequently, \( f^* \) has a continuous extension \( f^*_r \), from \( l_0^0 \) to \( l_r \). In fact, we can prove

**Theorem B.** If \( r \in I(\phi) \) and \( f \in W_p(J) \), then \( f^*_r = f(T_r) \in \mathfrak{C}_r \).

**Second application.** From now on, \( \{ \Phi_n : n \in a \} \) will be the trigonometric system; \( a \) is now the integer group and \( J = [0, 1] \). Originally proved by Stečkin in the case \( p = 1 \), property (i) was discovered by Hirschman [3]; his proof is based on Stečkin's result. Theorem B is also valid in the present context, the operator \( T \) being now the Hilbert transformation defined for all \( x \) in \( l_2 \) by the relation

\[ (Tx)_k = \sum_{k \in a} x_k \cdot \frac{i}{2\pi(n - k)} \quad (k \neq n). \]
for each $n$ in $a$. On the strength of Theorem A, we can prove Theorem B directly from the following well-known property: $|T|_s \neq \infty$ whenever $1 < s < \infty$.

**Theorem D.** Let $T_r$ be the unitary shift operator defined (for all $x$ in $l_r$) by the relation $T_r x = \{n \mapsto x_{n+1}\}$; there exists a function $E_r$ on $J$ to $\mathbb{C}$, such that

$$T_r = (E_r) \int e^{-2\pi i \lambda \cdot dE_r(\lambda)},$$

although $E_r$ is not of bounded variation.$^3$

5. **Hölder-type inequalities and the variation-norm.** We now return to the general setting of §3; once again, $(a, \alpha, \mu)$ is an arbitrary measure space, and the integrator $E$ is a resolution of the identity satisfying (v). If $x, y \in L^0(a, \alpha, \mu)$, then the relation

$$E_{x,y}(\lambda) = \int_a y \cdot E(\lambda) x \cdot d\mu \quad (\lambda \in J)$$

defines a complex-valued function $E_{x,y}$. The variation-norm is defined as follows:

$$V_q(E) = \sup \{ V_q(E_{x,y}) : x \in U_r \text{ and } y \in U_{r'} \},$$

where $U_r = \{ z \in L^0(a, \alpha, \mu) : ||z||_r \leq 1 \}$ and $r' = r/(r-1)$. When $f \in D(J)$ and $r = 2$ it is easy to verify the familiar inequality

$$||f||_2 \leq V_1(E) \cdot ||f||_\infty,$$

where $||f||_\infty = \sup \{ ||f(\lambda)|| : \lambda \in J \}$.

Suppose $1 < p < \infty$ and $r \in I(p)$. Our approach involves establishing the existence of a number $q > 1$ such that $q^{-1} + p^{-1} > 1$ and

$$||f||_q \leq c_{r,p} \cdot V_q(E) \cdot (||f||_\infty + V_p(f)) < \infty$$

(where $c_{r,p}$ is independent of $f$ and $E$), for each $f$ in $W_p(J)$. This is closely related to a theorem of Love and L. C. Young [5]; in fact, their work is based on the same inequality$^4$ that we use to prove (ii*).

6. **A convexity theorem for the variation-norm.** With (3) as a starting-point, it is easy to define an extension $F$ of the function

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$^3$ Due to L. C. Young [8]. The articles by Love and Young involve only scalar-valued functions.

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\{(q, r)\rightarrow V_q(E),\}\) such that \(F(\alpha^{-1}, \beta^{-1})\) is a convex function of \((\alpha, \beta)\) in the rectangle \(0 \leq \alpha, \beta \leq 1\). This fact plays a basic role in proving the results that have been presented.

7. **Remarks.** For more details concerning the Banach space \(W_p(J)\), see [5]. If \(p = 1\), then \(V_p(f)\) is the total variation of \(f\). The class \(W_p(J)\) becomes a Banach algebra under pointwise multiplication (and under the norm \(\{f \mapsto \|f\|_\infty + V_p(f)\}\)). The article [5] deals with continuous linear functionals on \(W_p(J)\).

**References**