THE MANIFOLD SMOOTHING PROBLEM

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The Schoenflies Theorem in $n$ dimensions has been proved by both Marston Morse [4] and Morton Brown [1] subject to the shell hypothesis [4]. Morse's proof leads to $C^m$-diffeomorphisms. We now prove the following Schoenflies Theorem for polyhedra without the shell hypothesis.

**Theorem 1.** Let $P^{n-1}$ be a combinatorial $(n-1)$-sphere in a euclidean $n$-space $E^n$, and let $N$ be an arbitrary neighborhood of $P^{n-1}$. Then $E^n$ can be mapped onto itself by a homeomorphism $h$ which is a $C^\infty$-diffeomorphism on $E^n - N$ and which maps $P^{n-1}$ onto a euclidean $(n-1)$-sphere $S^{n-1}$.

The proof commences with a modification of a procedure due to H. Noguchi [5] yielding an $\varepsilon$-isotopy of $E^n$ carrying $P^{n-1}$, on $D^n$, into a polyhedron $Q^{n-1}$, admitting a transverse vector field. A neighborhood of $Q^{n-1}$ is fibred by $C^0$-$(n-1)$-spheres, which permits a completion of the proof with the aid of Morse's methods [4]. His exceptional interior point can be relegated to $N$. The proof is inductive, requiring a partial assumption of Theorem 1 in the next lower dimension.

**Corollary.** Given a $\delta > 0$, $E^n$ admits a $\delta$-isotopy $h_t$ $(0 \leq t \leq 1)$ such that (1) $h_t$ is the identity on the unbounded component of $E^n - N$, (2) $h_t(P^{n-1}) \subset D^n$ $(t > 0)$ and (3) $h_t(P^{n-1})$ is a $C^\infty$-$(n-1)$-sphere $(t > \delta)$.

We will call a combinatorial $n$-manifold smoothable or nonsmoothable according as it is or is not compatible with a differentiable structure. The known nonsmoothable manifolds include a $K^8$ due to Milnor [3] and a $K^{10}$ due to Kervaire [2]. The latter is strongly nonsmoothable, in the sense that the topological manifold it covers, $M^{10} = |K^{10}|$, can not carry a differentiable structure, either compatible or incompatible with $K^{10}$.

A piecewise differentiable imbedding of a $K^n$ in a differentiable $n$-manifold $M^n$ means a homeomorphism $h: K^n \to M^n$, where $h$ is differentiable of maximal rank on each closed simplex of $K^n$.

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THEOREM 2. A combinatorial n-manifold $K^n$ without boundary is smoothable if and only if $K^n$ admits a piecewise differentiable imbedding $h$ into a differentiable $M^{n+1}$.

The necessity of the condition is easy to prove. The sufficiency proof commences with an $h: K^n \rightarrow M^{n+1}$ restricted, as in the proof of Theorem 1, so that $h(K^n)$ admits a transverse vector field on $M^{n+1}$. Let $M^{n+1}$ be represented as a differentiable submanifold of an $E^{n+r}$. With the aid of a potential function, equipotential $(n+r-1)$-manifolds surrounding $h(K^n)$ in $E^{n+r}$ can be defined [6]. If $h(K^n)$ is two-sided in $M^{n+1}$, the intersection $V^{n+r-1} \cap M^{n+1}$ with $M^{n+1}$ of an equipotential sufficiently near $h(K^n)$ falls into two components, $V_1^n$ and $V_2^n$, each of which is differentiable and homeomorphic to $K^n$. If $h(K^n)$ is one-sided in $M^{n+1}$, points can be so identified in pairs on $V^{n+r-1} \cap M^{n+1}$ as to obtain a differentiable homeomorph of $h(K^n)$.

COROLLARY. The $K^8$ of Milnor and $K^{10}$ of Kervaire do not admit piecewise differentiable imbeddings in differentiable 9-manifolds and 11-manifolds respectively.

THEOREM 3. If there exists a nonsmoothable $K^m$ without boundary, then there is a nonsmoothable $K^n$ without boundary for each $n > m$.

In particular, $K^m \times S^1$ where $S^1$ is a circle, is nonsmoothable, for its smoothability would imply that of $K^m$, by Theorem 2. Thus, all the manifolds $K^8 \times S^1 \times \cdots \times S^1$ and $K^{10} \times S^1 \times \cdots \times S^1$ are nonsmoothable, for Milnor's $K^8$ and Kervaire's $K^{10}$.

The invariants used by Milnor and Kervaire are thus freed from the dimensions for which they were defined. They are imbeddability as well as smoothability criteria.

REFERENCES


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