TOPOLOGICAL EQUIVALENCE OF A BANACH SPACE
WITH ITS UNIT CELL

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Several years ago [8] we proved that Hilbert space is homeomorphic with both its unit sphere \( \{ x : \| x \| = 1 \} \) and its unit cell \( \{ x : \| x \| \leq 1 \} \). Later [9] we showed that in every infinite-dimensional normed linear space, the unit sphere is homeomorphic with a (closed) hyperplane and the unit cell with a closed halfspace. It seems probable that every infinite-dimensional normed linear space is homeomorphic with both its unit sphere and its unit cell, but the question is unsettled even for Banach spaces. Corson [4] has recently proved that every \( \mathbb{R}^n \)-dimensional normed linear space is homeomorphic with its unit cell. In the present note, we establish the same result for a class of infinite-dimensional Banach spaces which is believed to include all such spaces. It is proved to include every infinite-dimensional Banach space which is reflexive, or admits an unconditional basis, or is a separable conjugate space, or is a space \( CM \) of all bounded continuous real-valued functions on a metric space \( M \).

We employ the following tools:

1. If \( E \) and \( F \) are Banach spaces and \( u \) is a continuous linear transformation of \( E \) onto \( F \), then there exist a constant \( m \in \mathbb{R}_+ \infty \) and a continuous mapping \( v \) of \( F \) into \( E \) such that \( uvx = x, vrx = rvx \), and \( \| vx \| \leq m \| x \| \) for all \( x \in F \) and \( r \in \mathbb{R} \) (the real number space). If \( G \) is the kernel of \( u \) and \( h = \{(uy, vuy) : y \in E \} \), then \( h \) is a homeomorphism of \( E \) onto \( F \times G \). Let \( \| (p, q) \| = \max \{ \| p \|, \| q \| \} \) for all \( (p, q) \in F \times G \), and let \( \xi y = \frac{\| y \|}{\| hy \|} hy \) for all \( y \in E \). Then \( \xi \) is a homeomorphism of \( E \) onto \( F \times G \) which carries the unit cell of \( E \) onto that of \( F \times G \).

2. If \( S \) is a closed linear subspace of a Banach space \( E \), then \( E \) is homeomorphic with the product space \( (E/S) \times S \) and the unit cell of \( E \) is homeomorphic with the unit cell of this product space (with respect to any norm compatible with the product topology).

3. In each infinite-dimensional normed linear space, the unit cell is homeomorphic with a closed halfspace.

4. If \( Q \) is an open halfspace in an infinite-dimensional normed linear space and \( p \) is a point in the boundary of \( Q \), then \( Q \cup \{ p \} \) is homeomorphic with \( Q \).

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For each \( f \in L^2[0, \infty) \), let the function \( f_t \) be defined as follows: 
\[ f_t(x) = \begin{cases} 
 f(x) & \text{for } x \in [0, 1], \\
 f(x + t - 1) & \text{for } x \in [1, \infty). 
\end{cases} \]
Then with \( \eta(f, t) = (f_t, t) \), the transformation \( \eta \) is a homeomorphism of \( L^2[0, \infty) \) onto \( L^2[0, 1] \) \( \cup \{x : 0 \leq x \leq 1\} \).

The existence of \( v \) and \( m \) as described in (1) follows from a theorem of Bartle and Graves [1, p. 404] (see also Michael [13]). It is easily verified that \( h \) is a homeomorphism \( [0, 1] \), and homogeneity of \( h \) follows from that of \( u \) and \( v \). Thus the transformation \( \xi \) is also homogeneous. To complete the proof of (1) it suffices to observe that 
\[ (1 + m)^{-\frac{1}{2}} \left\| y \right\| \leq \left\| hy \right\| \leq (m\|u\| + 1)\|y\| \]
for all \( y \in E \). Proposition (2) results from applying (1) to the canonical mapping \( u \) of \( E \) onto \( E/S \).

The result (3) appears in [9]. For (5), see page 29 of [8]. A theorem much stronger than (4) is proved on pages 12–28 of [8]. When the space is nonreflexive or is an \((l^p)\) space, (4) is explicitly a corollary of (3.3) on page 27 of [8]. In the general case, it follows from the reasoning (though not explicitly from any statement) in [8]. Also, a proof of (4) is outlined in [11].

A normed linear space \( J \) will be called compressible provided the space \( J \times [0, 1] \) is homeomorphic with the space \( (J \times [0, 1]) \cup (W \times \{0\}) \) for some closed linear subspace \( W \) of infinite deficiency in \( J \). (We see by (5) that Hilbert space is compressible.) A space is \( h\)-compressible provided it is homeomorphic with some compressible normed linear space.

**Theorem.** If a Banach space \( B \) admits a continuous linear transformation onto a Banach space \( E \) which contains an \( h\)-compressible closed linear proper subspace \( S \), then \( B \) is homeomorphic with the unit cell of \( B \).

**Proof.** Let \( G \) denote the kernel of the continuous linear transformation of \( B \) onto \( E \). By (1), \( B \) is homeomorphic with the product space \( P = E \times G \) and the unit cell of \( B \) is homeomorphic with the unit cell \( U \) of \( P \). To establish the theorem, it suffices to show that \( P \) is homeomorphic with \( U \). Since \( S \) is a closed linear proper subspace of \( E \), the subspace \( T = S \times \{0\} \) must be in a closed hyperplane \( V \) in \( P \). The unit cell \( U \) of \( P \) is homeomorphic with \( V \times [0, 1] \) by (3), and \( V \) is homeomorphic with \( (V/T) \times T \) by (2), so \( U \) is homeomorphic with \( (V/T) \times (T \times [0, 1]) \). Clearly \( P \) itself is homeomorphic with \( V \times [0, 1] \) and hence with \( (V/T) \times (T \times [0, 1]) \), so to complete the proof it suffices to show that \( T \times [0, 1] \) is homeomorphic with \( T \times [0, 1] \). Since \( T \) is \( h\)-compressible, there exist a Banach space \( J \) homeomorphic with \( T \) and a subspace \( W \) of infinite deficiency in \( J \) such that
$J \times [0, 1]$ is homeomorphic with $(J \times ]0, 1[) \cup (W \times \{0\})$. Let $u$ denote the canonical mapping of $J$ onto $J/W$ and then let $v$ and $h$ be as in (1) above. Then $h$ is a homeomorphism of $J$ onto $(J/W) \times W$, and since $hw = (\theta, v\theta - w)$ for all $w \in W$ (where $\theta$ is the neutral element of $J/W$), it follows that $hW = \{\theta\} \times W$. Consequently the space $(J \times ]0, 1[) \cup (W \times \{0\})$ is homeomorphic with

$$(J/W) \times W \times ]0, 1[ \cup \{\theta\} \times W \times \{0\},$$

which in turn is homeomorphic with

$$W \times ((J/W) \times ]0, 1[ \cup \{0\} \times \{0\}).$$

Since $J/W$ is infinite-dimensional, it follows by (4) that the set above is homeomorphic with

$$W \times ((J/W) \times ]0, 1[ \cup \{0\} \times \{0\}),$$

and hence with $J \times ]0, 1[$. Reviewing the information now assembled, we see that $T \times [0, 1[$ is homeomorphic with $T \times ]0, 1[$, and hence that $U$ is homeomorphic with $P$. This completes the proof of the theorem.

**Corollary.** If an infinite-dimensional Banach space $B$ satisfies at least one of the following conditions, then $B$ is homeomorphic with its unit cell:

(a) $B$ is reflexive;
(b) $B$ is a linear subspace of a Banach space which admits an unconditional basis;
(c) $B$ is a norm-separable $w^*$-closed linear subspace of a conjugate space;
(d) $B$ is the space $CN$ of all bounded continuous real-valued functions on a normal space $N$ which contains a closed infinite metrizable subset.

**Proof.** In view of the theorem and the fact (by (5)) that Hilbert space is compressible, it suffices in each case to produce a continuous linear transformation of $B$ onto a Banach space $E$ which contains a closed linear proper subspace $S$ which is homeomorphic with Hilbert space. When $B$ is reflexive, let $E = B$ and let $S$ be an infinite-dimensional separable closed linear proper subspace of $E$. Then $S$ is reflexive and hence (by a theorem of Kadec [7]) homeomorphic with Hilbert space.

If $B$ is a subspace of a space which admits an unconditional basis, a theorem of James [5] and Bessaga and Pelczyński [2] asserts that either $B$ is reflexive or some linear subspace of $B$ is linearly homeomorphic with the space $(l)$ or the space $(c_0)$. But the latter two spaces
are known to be homeomorphic with Hilbert space (by results of Mazur [12] and Kadeč [6]) and the desired conclusion follows.

Now suppose $B$ is a separable conjugate space or, more generally, that $B$ is a norm-separable $w^*$-closed linear subspace of a conjugate Banach space $L^*$. Let $f \in B \sim \{0\}$, $x \in L$ with $fx = 1$, and $S = \{g \in E : gx = 0\}$. Then $S$ is a $w^*$-closed linear proper subspace of $B$, and must be homeomorphic with Hilbert space by a theorem in [10]. Consequently, $B$ is homeomorphic with its unit cell.

Finally, let $B$ and $N$ be as in (d). Then there is a countably infinite closed subset $Z$ of $N$ which consists of either a discrete set or a convergent sequence together with its limit point. For each $\phi \in CN$ let $u\phi = \phi|Z \subset CZ$. Then $u$ is a continuous linear transformation of $CN$ onto $CZ$, and $CZ$ is equivalent to either the space $(m)$ or the space $(c_0)$. In either case, $CZ$ has the $h$-compressible space $(c_0)$ as a closed linear proper subspace, and the desired conclusion follows upon applying the theorem.

Note that the topological equivalence of every infinite-dimensional Banach space with its unit cell would be implied by the generally expected affirmative answer to the following question: Are all infinite-dimensional separable Banach spaces homeomorphic? Recent results on this problem have been obtained by Bessaga and Pełczyński [3].

At least for reflexive spaces, the corollary above can be significantly improved. The method is that of [8, pp. 30-31] in conjunction with the above techniques and the result is as follows:

**Theorem.** Suppose $E$ is an infinite-dimensional reflexive Banach space and $C$ is a closed convex subset of $E$ which has nonempty interior. Then $C$ is homeomorphic with $E$ and the boundary of $C$ is homeomorphic with $E$ or with $E \times S^n$ for some finite $n$ and $n$-sphere $S^n$.

The following problems seem worthy of mention: Are all infinite-dimensional separable Banach spaces $h$-compressible? (An affirmative answer implies that every infinite-dimensional Banach space is homeomorphic with its unit cell.) Are all infinite-dimensional Banach spaces compressible? Are $\aleph_\alpha$-dimensional normed linear spaces compressible? Note that for Hilbert space, the compressibility was achieved by means of a continuous family of affine homeomorphisms. How generally is this possible?

**References**


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